Lecture 1

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Lecture plan

- **Lecture 1: introduction, compression spaces, primitive spaces.**
- Lecture 2: spaces of matrices of constant and bounded rank, sheaves and vector bundles; dimension bounds.
- Lectures 3 & 4: the cases of symmetric and skew-symmetric matrices; applications to differential geometry and PDEs.
- Lecture 5: applications to numerical analysis: uniform determinantal representations and compression spaces.

Introduction

Definition (Spaces of matrices of constant and bounded rank)

Let V and W be two complex vector spaces, and consider a $d+1$-dimensional vector subspace

$$A \subseteq \mathrm{Hom}(V, W) \simeq V^{\vee} \otimes W.$$

A is a space of matrices of constant rank (respectively bounded rank) if all its non-zero elements have the same rank r (respectively rank $\leq r$).

Remark. Given a complex vector space V, we denote by $V^{\vee} = \mathrm{Hom}(V, \mathbb{C})$ its dual, and we fix a determinant form so that $V \simeq V^{\vee}$. Most of what we will see holds over any field $\boldsymbol{k} = \overline{\boldsymbol{k}}$, $char(\boldsymbol{k}) \neq 2$.

The interest for this kind of matrices, founded on classical work, bears to different contexts: linear algebra, algebraic geometry (vector bundles, varieties with degenerate dual), differential geometry, PDEs, numerical analysis...

Among the main questions, often still open, we identify **three main goals**:

- Search for maximal value of $\dim A$, and for relations among the values of the parameters $\dim V$, $\dim W$, constant or bounded rank r, $\dim A$.
- Classification of spaces for fixed values of the parameters.
- Construction of explicit examples.

<u>Remark.</u> Recall the Segre variety

$$Seg = Seg(\mathbb{P}V^\vee \times \mathbb{P}W) \hookrightarrow \mathbb{P}(V^\vee \otimes W),$$

that, being defined by the 2×2 minors of a generic matrix, corresponds to the variety of rank 1 matrices inside $\mathbb{P}(V^\vee \otimes W)$.

Its r-th secant variety $\sigma_r(Seg)$ is by definition the closure of the union of linear spans of all the r-tuples of independent points lying on Seg. It corresponds to matrices of rank at most r, and its singular locus is the variety $\sigma_{r-1}(Seg)$ of matrices of rank at most $r-1$.

Therefore a space of matrices of bounded or constant rank r is a linear space contained in the secant variety $\sigma_r(Seg)$, or in the stratum $\sigma_r \setminus \sigma_{r-1}$ respectively.

When one studies these vector spaces, algebraic geometry appears as a natural tool, as perhaps first observed in

[J. Sylvester, *On the dimension of spaces of linear transformations satisfying rank conditions*, Linear Algebra and its Applications, 78 (1986)].

For instance, vector bundles and their characteristic classes prove to be very useful: we will elaborate on this in the next lecture.

In this lecture we study the classification of spaces of low (that is, ≤ 3) bounded rank, following the paper

[D. Eisenbud and J. Harris *Vector spaces of matrices of low rank*, Advances in Mathematics, 70 (1988)].

Some of these results had already appeared in

[M. D. Atkinson, *Primitive spaces of matrices of bounded rank, II*, Journal of the Australian Mathematical Society, 34 (1983)],

but the point of view in Eisenbud-Harris is slightly more general and uses interesting algebraic geometry tools.

In fact the description of these spaces for $d + 1 = \dim A \leq 2$ (pencils of linear forms) is classical, see for example

[F.R. Gantmacher, *The Theory of Matrices Vol. 2*, Chelsea, Publishing Company, New York 1959].

Eisenbud-Harris' paper

As usual when trying to classify infinite families of mathematical objects, we first need a notion of "equivalence".

Definition

Two spaces of matrices $A, A' \subseteq \mathrm{Hom}(V, W)$ are called equivalent if they correspond under a change of bases.

Remark. In the definition above there is an action of a group taken into account! More precisely, if $\dim V = m$ and $\dim W = n$, then the group $G = \mathrm{GL}_n \times \mathrm{GL}_m$ acts on A by gAh^{-1} for any pair $(g, h) \in G$.

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Classification of spaces of matrices of bounded rank 1

The description of such spaces is classical; if all elements of A have rank ≤ 1, they must necessarily have

- either a common kernel $V' \subset V$ of codimension 1, so that A is equivalent to a subspace of the space of matrices of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * \end{pmatrix},$$

- or else a common image $W' \subset W$ of dimension 1, so that A is equivalent to a subspace of the space of matrices of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ * & * & \cdots & * \end{pmatrix}.$$

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Both these situations occurring for bounded rank 1 can be generalized to bounded rank r:

- the first one corresponds to the case when all elements of A have kernels containing a fixed subspace $V' \subset V$ of codimension r.
- Similarly, the second situation corresponds to all elements of A having image contained in a fixed r-dimensional subspace $W' \subset W$.

Clearly these cases above are "not interesting" in our classification: this motivates the following

Definition

A space of matrices A is nondegenerate if the kernels of the elements of A intersect in $\{0_V\}$, and the images of the elements of A generate W.

From now on, we will **assume that A is nondegenerate**: in other words, we don't consider spaces of matrices equivalent to ones whose elements all have rows or columns of zeros in common.

There is a different and more interesting generalization of the situations described above.

Definition (Compression spaces)

If there exist subspaces $V' \subset V$ of codimension r_1 and $W' \subset W$ of dimension r_2 such that

1. $r = r_1 + r_2$, and
2. every element of A maps V' into W',

then A is called a compression space.

In fact if the second condition holds and an element $\phi \in A$ map V' into W', then necessarily

$$r \leq r_1 + r_2;$$

when equality holds, the elements of A "compress" V' into W'.

Example

When $r = 2$, taking $r_1 = r_2 = 1$ we get a compression space equivalent to the vector space of matrices of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * \\ * & * & \cdots & * & * \end{pmatrix}.$$

Example

In general, a compression space of bounded rank r is equivalent to a space of $(\dim V) \times (\dim W)$ matrices having a common $v' \times w'$ block of zeros with $(\dim V - v') + (\dim W - w') = r$, the largest possible value.

As a corollary of the description of all spaces of matrices of rank ≤ 1 we get:

Theorem of classification of spaces of matrices of bounded rank 1

Every space of matrices of bounded rank 1 is a compression space.

⚠ This is only true for $r = 1$! Indeed, the space of 3×3 skew-symmetric matrices is a space of bounded rank 2 that is NOT a compression space.

There is one more uninteresting case that we need to rule out before we go on with our classification.

Let $V' \subset V$ of codimension r_1 and $W' \subset W$ of dimension r_2 be two subspaces, and consider $A' \subset \mathrm{Hom}(V', W/W')$ a space of matrices of bounded rank r'; then the space $A \subset \mathrm{Hom}(V, W)$ of matrices inducing the elements of A' via the composition:

$$V' \hookrightarrow V \to W \twoheadrightarrow W/W'$$

has rank bounded above by

$$r' + r_1 + r_2.$$

Let us call $\pi : \mathrm{Hom}(V, W) \twoheadrightarrow \mathrm{Hom}(V', W/W')$ this projection.

In other words, **from a space of matrices of a given bounded rank, one can cook up matrices of higher rank by adding some rows or columns of arbitrary entries**.

This motivates the following

Definition

- A space of matrices $A \subset \mathrm{Hom}(V, W)$ is primitive if there do not exist subspaces $V' \subset V$ and $W' \subset W$ with $(V', W') \neq (V, W)$ such that the upper bound of the rank of A is the same as the one of $\pi^{-1}(\pi(A))$.
- If A is not primitive, there exist subspaces V' and W' satisfying the condition above such that $A' = \pi(A)$ is primitive; A' is called a primitive part of A. (Which, in general, is not unique.)

Remark that **a compression space is exactly a space whose primitive part is zero**; in particular, there are no primitive spaces of bounded rank 1.

A projection map π corresponds, in terms of suitable bases, to taking submatrices. Thus A fails to be primitive if A is equivalent to a vector space of matrices in such a way that some space of submatrices A accounts entirely for the low rank of A; that is, A is a subspace of a vector space $\tilde{A}$ of matrices having the same rank as A and looking like

$$\tilde{A} = \begin{bmatrix} * & * & * & * & * & * & * \\ * & * & & & & & \\ * & * & & & A' & & \\ * & * & & & & & \end{bmatrix}$$

Examples/exercises:

- Show that the space of skew-symmetric 3×3 matrices is primitive.
- Show that the space
$$\begin{pmatrix} d & 0 & 0 & 0 \\ 0 & c & d & 0 \\ 0 & -b & 0 & d \\ -a & 0 & -b & -c \end{pmatrix}$$
is not primitive (because the rank of the family corresponding to the last three columns is 2).

Theorem of classification of spaces of matrices of bounded rank 2

A space of matrices of bounded rank 2 is either a compression space or is primitive, in which case it is the space of 3×3 skew-symmetric matrices.

⚠ Eisenbud-Harris' proofs for this and the next results relies on the theory of sheaves and vector bundles and their syzygies; we will see some ideas tomorrow.

In bounded rank 3 there are two complications: projections of the following space appear:

$$\begin{pmatrix} b & c & d & 0 & 0 & 0 \\ -a & 0 & 0 & c & d & 0 \\ 0 & -a & 0 & -b & 0 & d \\ 0 & 0 & -a & 0 & -b & -c \end{pmatrix},$$

and there are imprimitive spaces which are not compression spaces.

Theorem of classification of spaces of matrices of bounded rank 3

A nondegenerate space of matrices of bounded rank 3 is either a compression space, or primitive, or it has bounded rank 3 and its primitive part is the space of 3×3 skew-symmetric matrices, so that it is of the form

$$\begin{pmatrix} 0 & a & b & * \\ -a & 0 & c & * \\ -b & -c & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

or its transpose.

The primitive spaces themselves can be classified:

Theorem of classification of primitive spaces of bounded rank 3

A primitive space of matrices of bounded rank 3 is equivalent either to

$$\begin{pmatrix} b & c & d & 0 & 0 & 0 \\ -a & 0 & 0 & c & d & 0 \\ 0 & -a & 0 & -b & 0 & d \\ 0 & 0 & -a & 0 & -b & -c \end{pmatrix} \qquad (\star)$$

or its transpose or to one of the following four projections of $(\star)$ and their transposes, which are themselves primitive and pairwise inequivalent:

$$\begin{pmatrix} -b & -c & -d & 0 & 0 \\ a & 0 & 0 & -c & -d \\ -d & a & 0 & b & 0 \\ c & 0 & a & 0 & b \end{pmatrix}, \quad \begin{pmatrix} -b & -c & -d & 0 & 0 \\ a & 0 & 0 & -c & -d \\ 0 & a & 0 & b & 0 \\ 0 & 0 & a & 0 & b \end{pmatrix},$$

$$\begin{pmatrix} c & d & 0 & 0 \\ 0 & 0 & c & d \\ -a & 0 & -b & 0 \\ 0 & -a & 0 & -b \end{pmatrix}, \quad \begin{pmatrix} -b & -d & 0 & 0 \\ a & 0 & -c & -d \\ -d & 0 & b & 0 \\ c & a & 0 & b \end{pmatrix}.$$

To conclude, remark that the classification for rank 2 and 3 gives in particular a complete classification of inclusion-maximal singular matrix spaces of size $2, 3, 4$ (including degenerate cases).

For $n = 2$, every singular matrix space is a compression space, hence conjugate to a subspace of one of the two spaces

$$\left\{ \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \right\}, \qquad \left\{ \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \right\}.$$

For $n = 3$, there are four conjugacy classes of inclusion-maximal singular matrix spaces, represented by the three maximal compression spaces

$$\left\{ \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix} \right\},$$

together with the space of skew-symmetric 3×3 matrices.

Unfortunately for large size it seems impossible to classify these spaces.

For $n = 4$, there are still finitely many (namely, 10) conjugacy classes of inclusion-maximal singular matrix spaces (see [Eisenbud-Harris] and

[P. Fillmore, C. Laurie, and H. Radjavi, *On matrix spaces with zero determinant*, Linear and Multilinear Algebra, 18 (1985)]),

but this is **not true** for $n \geq 5$.

There are other evidences: for instance, for infinitely many n there exists a maximal singular matrix space in $K^{n \times n}$ of constant dimension 8, at least if we assume that the field K has characteristic 0, see

[J. Draisma, *Small maximal spaces of non-invertible matrices*, Bulletin of the London Mathematical Society, 38 (2006)].