KTH ROYAL INSTITUTE
OF TECHNOLOGY

Algebraic Geometry with a view towards applications

Sandra Di Rocco, ICTP Trieste

Plan for this course

- Lecture I: Algebraic modelling (Kinematics)
- Lecture II: Sampling algebraic varieties: the reach.
- Lecture III: Projective embeddings and Polar classes (classical theory)
- **Lecture IV: The Euclidian Distance Degree**
- Lecture V: Bottle Neck degree from classical geometry (back to sampling)

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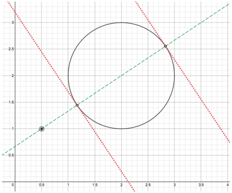

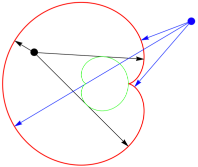

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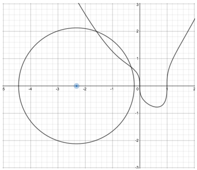

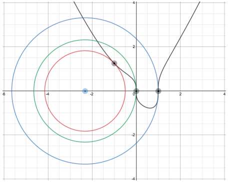

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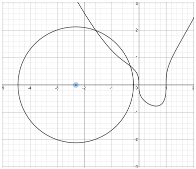

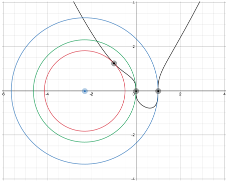

C is a conic: $3x3$ matrix $M_1 = (c_{ij})$
the circle given by the the $3x3$ symmetric matrix
$M_2 = M(u, r)$.
$2x3x3$ hyperderminant: $H(c_{ij}, u, r) = 0$ of degree 4 in r^2.

Theorem (Cayley)

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The key tool is the use of *Schläfli decomposition*

$$MD(A_1, A_2) = Hyperdet([M_1, M_2]) = Disc_t(det(M_1 + tM_2)).$$

References

- R. Thomas, *Euclidean Distance Degree*, SIAM News October 2014
- J. Draisma,E. Horobet, G. Ottaviani, B. Sturmfels, R. Thomas, *The Euclidean Distance Degree of a variety*, Fo.C.M 2015.
- R. Piene, Polar Varieties Revisited, *Computer Algebra and Polynomials* Springer, 2015.

Polarity with respect to Q

- Let $Q = (\sum x_i^2 = 0) \subset \mathbb{P}^N$ be the *isotropic quadric*
- $p = (a_0, \dots, a_N) \in \mathbb{P}^N$, $p^\perp = (\sum \frac{\partial Q}{\partial x_i} \cdot a_i = 0) \in (\mathbb{P}^N)^*$.
- L linear of $dim = k$, $L^\perp = \cap_{p \in L} p^\perp$ has $dim = N - k - 1$.

$$x \perp y \text{ if and only if } x \in y^\perp \text{ i.e. } \sum x_i y_i = 0.$$

Observe that $x \in X$ is a critical point for $d_u(X)$ if and only if $u - x \in T_{X,x}^\perp$.

Projective case

(*) for technical reasons we assume that $X \cap Q = \varnothing$. Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension n.

$$EDD_u(X) = EDD_u(C(X))$$

where $C(X)$ is the affine cone of X in $\mathbb{C}^{N+1}$.
critical points $x \in X$ w.r.t $u \in \mathbb{P}^N$ satisfy:

$$rank \begin{bmatrix} u \\ x \\ J_x \end{bmatrix} < c + 1$$

Plane curve

Let $I(C) = (F)$, $F \in \mathbb{C}[x_0, x_1, x_2]$, of degree d.
Look for y such that

$$F(y) = \det \begin{bmatrix} u_0 & u_1 & u_2 \\ y_0 & y_1 & y_2 \\ \frac{\partial F}{\partial x_0}(y) & \frac{\partial F}{\partial x_1}(y) & \frac{\partial F}{\partial x_2}(y) \end{bmatrix} = 0$$

$EDD_u = d^2$.

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$p \in X$, $N_pX =< T_pX^{\perp}, p >\cong \mathbb{P}^{N-n}$ is called the *Euclidean Normal Space.*

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$$0 \to K \to \bigoplus_{0}^{N} \mathcal{O}_X \to J_1 \to 0$$

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$\mathbb{P}(E)$ is the **Euclidean Normal bundle**

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Finite general EDD

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$$EDD_u(X) = EDD(X)$$

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Under the previous assumptions:

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(*) True in much more generality!

Polar calculus: example

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Table:

	P_1	P_2	P_3	P_1P_2	P_1^2D	P_2D	P_1D^2	D^3
degree	8	12	16	24	32	24	32	32

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$EDD(X) = 40$ and $\beta(X) = ?$