Tensors determined by their singular tuples

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Eigentensors and singular tuples

Let $f \in Sym^d V$ and $q = x_0^2 + \cdots + x_m^2$ a quadratic form on V, the eigentensors of f are vectors $x \in V$ such that

$$f(x^{d-1}) = \lambda x,$$

for $\lambda \in \mathbb{C}$.

The eigenscheme $Eig(f)$ of eigentensors of f is given by the 2×2-minors of the matrix

$$\begin{bmatrix} \frac{\partial f}{\partial x_0} & \cdots & \frac{\partial f}{\partial x_m} \\ x_0 & \ldots & x_m \end{bmatrix}.$$

Let $T \in Sym^{d_1}V_1 \otimes \cdots \otimes Sym^{d_k}V_k$ and $q_i = x_0^2 + \cdots + x_{m_i}^2$ a quadratic form on V_i. The singular tuples of T are the tuples $(v_1, \ldots, v_n)$, such that each flattening

$$T_i : Sym^{d_1}V_1 \otimes \cdots \otimes Sym^{d_i - 1}V_i \otimes \cdots \otimes Sym^{d_k}V_k \to V_i$$

satisfies

$$T_i(v_1^{d_1} \otimes \cdots \otimes v_i^{d_i-1} \otimes \cdots \otimes v_k^{d_k}) = \lambda v_i.$$

The eigenscheme of T is the zero dimensional scheme $Eig(T)$ defined by the locus of singular tuples of T.

The eigenscheme as the zero locus of a bundle

Let $X = \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k$ be the Segre-Veronese variety of rank 1 tensors embedded with $\mathcal{O}(d_1, \ldots, d_k)$ in $\mathbb{P}(Sym^{d_1}V_1 \otimes \cdots \otimes Sym^{d_k}V_k)$. Let $\pi_l : X \to \mathbb{P}V_l$ be the projection on the l-th component, and let Q_l be the quotient bundle, whose fibers over a point $v_l \in V_l$ are $V_l/\langle v_l \rangle$. Let $\mathcal{E}_l = \pi_l^* Q_l \otimes \mathcal{O}(d_1, \ldots, d_l - 1, \ldots, d_l)$, we can construct the vector bundle

$$\mathcal{E} = \bigoplus_{l=1}^{k} \mathcal{E}_l.$$

A tensor $T \in \mathbb{P}(Sym^{d_1}V_1 \otimes \cdots \otimes Sym^{d_k}V_k)$ leads to a global section of $\mathcal{E}_l$ which over a point $v = (v_1, \ldots, v_k)$ is the map sending v to the natural pairing of f with $(v_1^{d_1}) \cdots (v_l^{d_l - 1}) \cdots (v_k^{d_k})$ modulo $\langle v_l \rangle$, that is a vector in $V_l/\langle v_l \rangle$.

In [4] it is proven that if we consider the section s_T associated to a general multisymmetric tensor T, then the zero locus $Z(s_T)$ is equal to the locus of singular tuples of T, that is $Z(s_T) = Eig(T)$.

Main problem and technique

The question we pose is: *given the singular tuples of a tensor T, which are all the tensors that admit such configuration of singular tuples?*

Consider the map

$$\tau : \mathbb{P}\big(Sym^{d_1}V_1 \otimes \cdots \otimes Sym^{d_k}V_k\big) \dashrightarrow \big[\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_k)\big]^{(ed_X)}$$

that for a general tensor $T \in Sym^{d_1}V_1 \otimes \cdots \otimes Sym^{d_k}V_k$ it associates its singular tuples locus $Eig(T)$.

Studying the map τ directly is difficult, thus we decompose τ in the following manner

$$\begin{array}{ccc} \mathbb{P}\big(Sym^{d_1}V_1 \otimes \cdots \otimes Sym^{d_k}V_k\big) & \overset{\tau}{\dashrightarrow} & \big[\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_k)\big]^{(ed_X)} \\ & \searrow^{\varphi} \quad \nearrow^{\psi} & \\ & \mathbb{P}(H^0(\mathcal{E})) & \end{array}$$

This gives a better description since $\varphi : T \mapsto s_T$ is a linear map and $\psi : s \mapsto Z(s)$ can be approached using the Koszul complex in a similar fashion as [3].

References

[1] Hirotachi Abo, Anna Seigal, and Bernd Sturmfels. "Eigenconfigurations of tensors". In: (2017).

[2] Valentina Beorchia, Francesco Galuppi, and Lorenzo Venturello. *Eigenschemes of ternary tensors*. 2020.

[3] Jan Draisma, Giorgio Ottaviani, and Alicia Tocino. "Best rank-k approximations for tensors: generalizing Eckart-Young". In: (2017).

[4] Shmuel Friedland and Giorgio Ottaviani. "The Number of Singular Vector Tuples and Uniqueness of Best Rank-One Approximation of Tensors". In: (2014).

Lemma: Let V be a vector space of dimension $m+1$, $q = x_0^2 + \cdots + x_m^2$ a quadratic form on V, and d a positive integer. If d is odd, the map $\varphi : Sym^d V \to H^0(Q(d-1))$ is injective. If d is even, φ has a 1-dimensional kernel, namely, $\ker \varphi = \langle q^{d/2} \rangle$.

Lemma: Let $f, g \in Sym^d V$ be two general polynomials such that $Z(s_f) = Z(s_g)$, $d \geq 3$. Then $s_f = \alpha s_g$ for some $\alpha \in \mathbb{C}^*$.

Theorem: Let V be a vector space of dimension $m+1$. Let $d \geq 3$ be an integer, and $f \in \mathbb{P}(Sym^d V)$ be a general polynomial. Let

$$\tau : \mathbb{P}\big(Sym^d V\big) \dashrightarrow \mathbb{P}V^{(ed_X)}, \; f \mapsto Eig(f)$$

be the map that associates f to its eigentensors locus $Eig(f)$. Then

$$\tau^{-1}(\tau(f)) = \begin{cases} [f], & \text{if } d \text{ is odd;} \\ \{[f + cq^{\frac{d}{2}}] | c \in \mathbb{C}\}, & \text{if } d \text{ is even.} \end{cases}$$

This result generalises the result obtained in [2] for symmetric tensors in 3 variables to any number of variables.

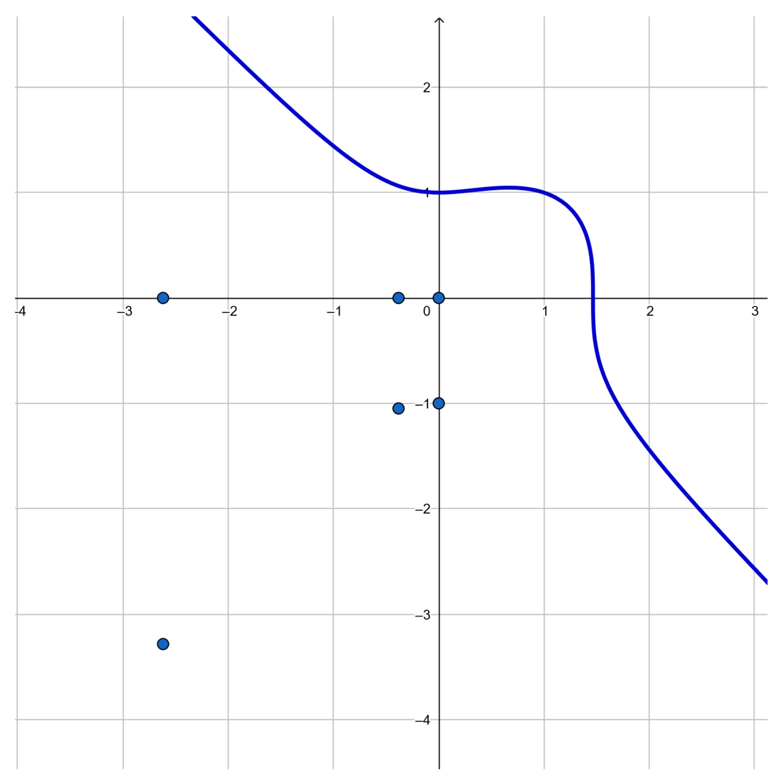

Fig. 1: The cubic curve $x^3 + y^3 - x^2 - 1$ and its real eigentensors.

Theorem: Let $V_1, \ldots, V_k$ be vector spaces of dimension $m_1 + 1, \ldots, m_k + 1$. Consider the map

$$\varphi : Sym^{d_1}V_1 \otimes \cdots \otimes Sym^{d_k}V_k \to H^0(\mathcal{E}).$$

Then φ is injective if at least one d_i is odd. In the case that all the d_i are even, we have that the kernel of φ is one dimensional and it is given by

$$\ker \varphi = \langle q_1^{\frac{d_1}{2}} \rangle \otimes \cdots \otimes \langle q_k^{\frac{d_k}{2}} \rangle$$

Theorem: Let $S, T \in Sym^{d_1}V_1 \otimes \cdots \otimes Sym^{d_k}V_k$ be two general tensors. Let $s, t \in \mathcal{E}$ be the sections coming from the tensors, S and T, and assume that $Z(s) = Z(t)$, then $s = \lambda t$, for $\lambda \in \mathbb{C}^*$.

Theorem: Let $V_1, \ldots, V_k$ be vector spaces of dimension $m_1+1, \ldots, m_k+1$. Let $d_1, \ldots, d_k$ be positive integers, and $T \in \mathbb{P}\big(Sym^{d_1}V_1 \otimes \cdots \otimes Sym^{d_k}V_k\big)$ be a general tensor. Let

$$\tau : \mathbb{P}\big(Sym^{d_1}V_1 \otimes \cdots \otimes Sym^{d_k}V_k\big) \dashrightarrow (\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k)^{(ed_X)}, T \mapsto Eig(T),$$

be the map that associates a tensor T to its singular tuples locus $Eig(T)$. If $k \geq 3$ and suppose that $m_l \leq \sum_{j \neq l} m_j$ whenever $d_l = 1$, and for $k = 2$ we include the hypothesis that $(d_1, d_2) \neq (1, 1)$, then

$$\tau^{-1}(\tau(T)) = \begin{cases} [T], & \text{if } d_i \text{ is odd for some } i; \\ \{[T + cq_1^{\frac{d_1}{2}} \otimes \cdots \otimes q_k^{\frac{d_k}{2}}] | c \in \mathbb{C}\}, & \text{if } d_l \text{ is even for all } l. \end{cases}$$

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