Schur apolarity

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Structured tensors

An additive decomposition of a structured tensor T is something like

$$T = c_1 T_1 + \cdots + c_r T_r, \ c_i \in \mathbb{C},$$

where T_i are called *tensors of structured rank* 1. Famous examples of structured tensors are the *symmetric* tensors $\mathrm{Sym}^d V$ and the *skew-symmetric* tensors $\bigwedge^d V$.

The apolarity theory make use of **apolarity actions** to study the symmetric or skew-symmetric rank

$$\mathrm{Sym}^d V \otimes \mathrm{Sym}^e V^* \longrightarrow \mathrm{Sym}^{d-e} V, \ 0 \le e \le d$$

$$\bigwedge^d V \otimes \bigwedge^e V^* \longrightarrow \bigwedge^{d-e} V, \ 0 \le e \le d \le n$$

Question

Does there exists an analogous apolarity theory for other irreducible representations of $SL(n)$?

Consider the Schur (Weyl) module $\mathbb{S}_\lambda V$ of $SL(n)$, where the symbol λ is a non-increasing seq. $(\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \ge \cdots \ge \lambda_k \ge 0$ and $k < n$. We have the natural inclusion

$$\mathbb{S}_\lambda V \subset \bigwedge^{\lambda'_1} V \otimes \cdots \otimes \bigwedge^{\lambda'_k} V =: \bigwedge_{\lambda'} V$$

where λ' is the conjugate partition to λ.

Skew-symmetric apolarity

Recall also the definition of the *skew-symmetric apolarity action*. Given $0 \le e \le d < n$, it is the composition of maps

$$\wedge^d V \otimes \wedge^e V^* \longrightarrow \wedge^{d-e} V \otimes \wedge^e V \otimes \wedge^e V^* \longrightarrow \wedge^{d-e} V.$$

Related works

- A. Iarrobino and V. Kanev, Power sums, Gorenstein algebras, and determinantal loci
- E. Arrondo, A. Bernardi, P. M. Marques and B. Mourrain, Skew-symmetric tensor decomposition
- J. M. Landsberg and G. Ottaviani, Equations for secant varieties of Veronese and other varieties

Basic definitions

In the following we reduce to work in the vector space $\mathbb{S}^\bullet V$. It is defined as the quotient

$$\mathrm{Sym}^\bullet(V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^{n-1} V)/I^\bullet$$

where $I^\bullet$ is the ideal generated by

$$(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_q) - \sum_{i=1}^{p} (v_1 \wedge \cdots \wedge w_1 \wedge \cdots \wedge v_p) \cdot (v_i \wedge w_2 \wedge \cdots \wedge w_q)$$

for any $p \ge q \ge 0$, known as *Plücker relations*.

Schur apolarity

In the following the notation $\mu \subset \lambda$ means that $\mu_i \le \lambda_i$ for every i.

Definition. The *Schur apolarity action* is the map

$$\varphi : \mathbb{S}^\bullet V \otimes \mathbb{S}^\bullet V^* \longrightarrow \mathbb{S}^\bullet V$$

such that when restricted to $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^*$ is

- the zero map if $\mu \not\subset \lambda$,
- otherwise it is the restriction of the map

$$\tilde{\varphi} : \bigwedge_{\lambda'} V \otimes \bigwedge_{\mu'} V^* \longrightarrow \bigwedge_{\lambda'/\mu'} V$$

given by products of $\wedge^{\lambda'_i} V \otimes \wedge^{\mu'_i} V \longrightarrow \wedge^{\lambda'_i - \mu'_i} V$.

Proposition. For λ and μ such that $\mu \subset \lambda$, the image of the Schur apolarity action is contained in $\mathbb{S}_{\lambda/\mu} V$.

Definition. For a fixed $t \in \mathbb{S}_\lambda V$, the induced map

$$\mathcal{C}_t^{\lambda,\mu} : \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_{\lambda/\mu} V$$

that sends g to $\varphi(f \otimes g)$ is called **catalecticant map of λ and μ.**

The geometry

The minimal orbit X inside $\mathbb{P}(\mathbb{S}_\lambda V)$ is a Flag variety $\mathbb{F}(k_1, \dots, k_s; V)$ of flags $V_1 \subset \cdots \subset V_s \subset V$, with $\dim V_i = k_i$, embedded with $\mathcal{O}(d_1, \dots, d_s)$. The points of X-rank 1 are something like

$$(v_1 \wedge \cdots \wedge v_{k_s})^{\otimes d_s} \otimes \cdots \otimes (v_1 \wedge \cdots \wedge v_{k_1})^{\otimes d_1}$$

representing the flag

$$\langle v_1, \dots, v_{k_1} \rangle \subset \cdots \subset \langle v_1, \dots, v_{k_s} \rangle.$$

The Schur apolarity lemma

Consider a point $p \in X$ of rank 1 as in the previous box and consider the flag of orthogonal spaces

$$V_s^\perp \subset \cdots \subset V_1^\perp \subset V^*.$$

Construct a subspace $I(p) \subset \mathbb{S}^\bullet V^*$ in the following way

- start with the spaces $V_s^\perp$, $\mathrm{Sym}^{d_s+1} V_{s-1}^\perp, \dots \mathrm{Sym}^{d_s+\cdots+d_2+1} V_1^\perp$,
- for any ν, the space $I(p) \cap \mathbb{S}_\nu V^*$ is obtained recursively via the maps of representations

$$\mathbb{S}_\lambda V^* \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_\nu V^*$$

restricted to $(I(p) \cap \mathbb{S}_\lambda V^*) \otimes \mathbb{S}_\mu V^*$. If $p_1, \dots, p_r$ are points of X-rank 1, define

$$I(p_1, \dots, p_r) := I(p_1) \cap \cdots \cap I(p_r).$$

Lemma: Let $f \in \mathbb{S}_\lambda V$ be any point and let $p_1, \dots, p_r \in \mathbb{S}_\lambda V$ be points of X-rank 1, where X is the minimal orbit inside $\mathbb{P}(\mathbb{S}_\lambda V)$. Then the following are equivalent:

- we may write $f = c_1 p_1 + \cdots + c_r p_r$ for some $c_i \in \mathbb{C}$,
- we have the inclusion $I(p_1, \dots, p_s) \subset f^\perp$, where this last one is the **apolar set to** f and is given by any element of $\mathbb{S}^\bullet V^*$ that kills f via the Schur apolarity action.

An example

Let $\lambda = (2,1)$ and $t = v_2 \wedge v_3 \otimes v_1 - v_1 \wedge v_2 \otimes v_3 \in \mathbb{S}_{(2,1)}\mathbb{C}^n$. The related variety is $X = \mathbb{F}(1,2;n)$ embedded with $\mathcal{O}(1,1)$.

Then t represents a flag $V_1 \subset V_2 \subset \mathbb{C}^n$, i.e. t has rank 1, if and only if $V_2^\perp \subset \ker \mathcal{C}_t^{(2,1),(1)}$. Since this last one has codimension 3 instead of 2, then t has not rank 1.

If $t = t_1 + t_2$, with $t_i \in X$, we must find at least a product of linear forms in $\ker \mathcal{C}_t^{(2,1),(2)}$. Since $(x_1 - x_3)(x_1 + x_3)$ belongs to this kernel, we may set the system for $a, b, c, d \in \mathbb{C}$

$$t = (a(v_1 - v_3) + bv_2) \wedge v_2 \otimes (a(v_1 - v_3) + bv_2) + \\ + (c(v_1 + v_3) + dv_2) \wedge v_2 \otimes (c(v_1 + v_3) + dv_2)$$

which has solution for $a = \frac{1}{2}$, $c = -\frac{1}{2}$ and $b = d = 0$.