Decomposition loci of tensors

Work in progress
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Denote by $\mathbb{T}_{n_1\cdots n_k}$ the cone of the rank-1 tensors in $\mathbb{C}^{n_1}\otimes\cdots\otimes\mathbb{C}^{n_k}$ and take a rank-r tensor $T\in\mathbb{C}^{n_1}\otimes\cdots\otimes\mathbb{C}^{n_k}$.

Decomposition locus of T

$$\mathcal{D}_T := \{P\in\mathbb{T}_{n_1\cdots n_k} \mid \exists P_2,\dots,P_r\in\mathbb{T}_{n_1\cdots n_k},\ T\in\langle P,P_2,\dots,P_r\rangle\}$$

Forbidden locus of T

$$\mathcal{F}_T := \mathbb{T}_{n_1\cdots n_k}\setminus\mathcal{D}_T$$

GOAL: Compute either $\mathcal{D}_T$ or $\mathcal{F}_T$.

Given a rank-r tensor T,
if $P\in\mathcal{F}_T$ then $r(T-\lambda P)\geq r$ for all $\lambda\in\mathbb{C}$.

Recall that...

The spaces of tensors $\mathbb{C}^{n_1}\otimes\cdots\otimes\mathbb{C}^{n_k}$ with a finite number of $GL_{n_1}\times\cdots\times GL_{n_k}$-orbits are

1. $\mathbb{C}^n\times\mathbb{C}^m$
2. $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^n$
3. $\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^n$

Matrices

Let $A\in M_n(\mathbb{C})$ be an invertible matrix.
We look for $P=uv^T$ such that

$$\det(A-\lambda uv^T)\neq 0 \text{ for all } \lambda\in\mathbb{C}.$$

$$\mathcal{F}_A=\left\{u,v\in\mathbb{C}^n \mid v^TA^{-1}u=0\right\}.$$

This covers the whole matrix case.

Rank-3 tensors in $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$

The normal form of a rank-3 tensor in $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ is $T=e_2\otimes e_1\otimes e_1+e_1\otimes e_2\otimes e_1+e_1\otimes e_1\otimes e_2$.

$$[T-\lambda P]\in\tau(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1) \iff P=e_1\otimes e_1\otimes e_1.$$

Therefore

$$\mathcal{F}_T=\{e_1\otimes e_1\otimes e_1\}.$$

Max-rank tensors in $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^4$

The normal form of a maximal rank tensor in $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^4$ is the rank-4 tensor $T=e_1\otimes(e_1\otimes e_1+e_2\otimes e_3)+e_2\otimes(e_1\otimes e_2+e_2\otimes e_4)$.

$$P=\begin{pmatrix}a_1\\a_2\end{pmatrix}\otimes\begin{pmatrix}b_1\\b_2\end{pmatrix}\otimes\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}\in\mathcal{F}_T$$

$$\Updownarrow$$

$$a_1b_1c_1+a_2b_1c_2+a_1b_2c_3+a_2b_2c_4=0.$$

Algorithm for tensors with $\mathrm{br(T)}=\mathrm{r(T)}=3$

Input: T with $r(T)=br(T)=3$.
Output: Decomposition locus of T.

- $P:=$ generic rank-1 tensor.
- $I:=\left(\begin{array}{c}3\times 3 \text{ minors of the}\\ \text{flattenings of } T-\lambda P\end{array}\right)$.
- $J:=$ (eliminate λ from I).
- $\tilde{\lambda}:=\lambda$ such that $T-\lambda P$ satisfies J.
  i.e. $T-\tilde{\lambda}P$ lies in the 2^{nd} sec. variety.
- Compute the concise tensor space X_{con} of $T-\tilde{\lambda}P$.
  - If $X_{con}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$, test membership with $\tau(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)$ and discard the corresponding values;
  - Else, the points finded will form the decomposition locus of T.

STAY TUNED
for other decomposition loci!

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