Rank additivity property and improved substitution method

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Introduction

Let A, B, and C be finitely dimensional vector spaces over $\Bbbk$. A *simple tensor* is an element of the tensor space $A \otimes B \otimes C$ which can be written as $a \otimes b \otimes c$ for some $a \in A$, $b \in B$, $c \in C$. The *rank of a tensor* $p \in A \otimes B \otimes C$ is the minimal number $R(p)$ of simple tensors needed, such that p can be expressed as a linear combination of them. In general, the higher the rank is, the more complicated p "tends" to be.

Fast matrix multiplication and motivation

The matrix multiplication is a bilinear map, thus we can interpret it as a three-way tensor

$$\mu_{a,b,c} \in (\mathcal{M}^{a \times b})^* \otimes (\mathcal{M}^{b \times c})^* \otimes \mathcal{M}^{a \times c}.$$

The minimal number of multiplications needed to calculate the product of two matrices is equal to $R(\mu_{a,b,c})$.

Given four arbitrary matrices $M' \in \mathcal{M}^{a' \times b'}$, $N' \in \mathcal{M}^{b' \times c'}$, $M'' \in \mathcal{M}^{a'' \times b''}$, $N'' \in \mathcal{M}^{b'' \times c''}$, suppose we want to calculate both products $M'N'$ and $M''N''$ simultaneously. What is the minimal number of multiplications needed to obtain the result? Is it equal to the sum of the ranks, i.e.

$$R(\mu_{a',b',c'} + \mu_{a'',b'',c''}) = R(\mu_{a',b',c'}) + R(\mu_{a'',b'',c''})?$$

Generalisation and history

More generally, if we are given two tensors in independent vector spaces, is the rank of their sum equal to the sum of their ranks? This problem was widely known as *Strassen's Conjecture* [9]. Important step in the topic was [4]. In [8] Y. Shitov shows a counterexample for Strassen's additivity conjecture for $A' = A'' = \ldots = C' = C'' = \mathbb{C}^n$, where $n \geq 450$ (see notation in the "Strassen's additivity problem" section), but it is not a constructive proof.

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Strassen's additivity problem

Suppose $A = A' \oplus A''$, $B = B' \oplus B''$, and $C = C' \oplus C''$, where all $A, \ldots, C''$ are finitely dimensional vector spaces. Let

$$p = p' + p'',$$

where $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$.

What are dimensions of $A', A'', \ldots, C''$ such that the *additivity of ranks* holds:

$$R(p) = R(p') + R(p'')?$$

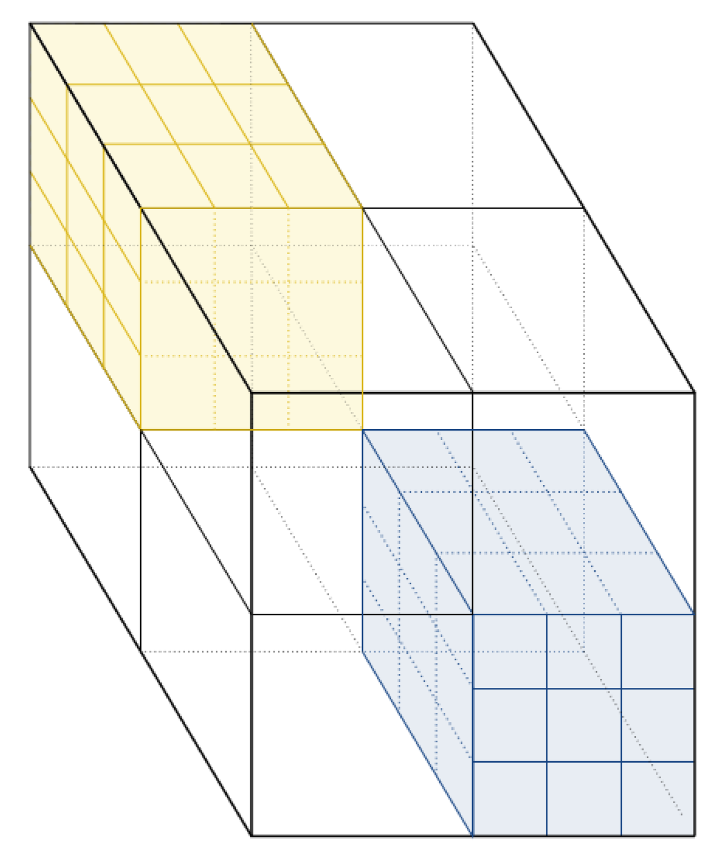

Figure 1: The tensor $p = p' + p'' \in (\Bbbk^3 \oplus \Bbbk^3) \otimes (\Bbbk^3 \oplus \Bbbk^3) \otimes (\Bbbk^3 \oplus \Bbbk^3)$.

Slice technique

A tensor $p \in A \otimes B \otimes C$ determines a linear map $p : A^* \to B \otimes C$. Consider the image $W = p(A^*) \subset B \otimes C$. The subspace W contains all the geometric information about p, in particular its rank, $R(p) = R(W)$.

Projections and notation

Let V be a linear space of dimension $R(p)$ spanned by rank one matrices such that $p(A^*) =: W = W' \oplus W'' \subset V \subset B \otimes C$. Furthermore, let V_{Seg} denote the set of rank one matrices of V, this is the intersection of V and subset of simple tensors in $B \otimes C$.

Consider the natural projection of spaces of matrices $\pi_{C'} : B \otimes C \to B \otimes C''$. Let $E' \subset B'$ be the minimal vector subspace such that $\pi_{C'}(V)$ is contained in $(E' \oplus B'') \otimes C''$. We define $F' \subset C'$, $F'' \subset C''$ and $E'' \subset B''$ similarly.

Main results

Additivity of rank holds for p' and p'', if at least one of the following condition holds:

- $R(p'') \leq \dim A'' + 2$ and p'' is not contained in $\tilde{A}'' \otimes B'' \otimes C''$ for any linear subspace $\tilde{A}'' \subsetneq A''$.
- $p'(A'^*)$ is a $(2,1)$-hook, i.e. $p'(A'^*)$ is a subset of $B' \otimes \Bbbk^1 + \Bbbk^2 \otimes C'$
- $\Bbbk = \mathbb{C}$ or $\Bbbk = \mathbb{R}$, and $\dim B'' \leq 3$, $\dim C'' \leq 3$.
- $\Bbbk = \mathbb{C}$ or $\Bbbk = \mathbb{R}$, and $R(p'') \leq 6$.
- $\Bbbk$ is s.t. $\max\{R(t) \mid t \in \Bbbk^3 \otimes \Bbbk^3 \otimes \Bbbk^3\} \leq 5$ and $R(p'') \leq 6$. For example $\Bbbk$ is algebraically closed of characteristic $\neq 2$.
- $\Bbbk = \mathbb{C}$ and dim. of $((A', B', C'), (A'', B'', C''))$ equal $((4,4,3),(4,4,3))$ or $((4,4,3),(4,3,4))$,

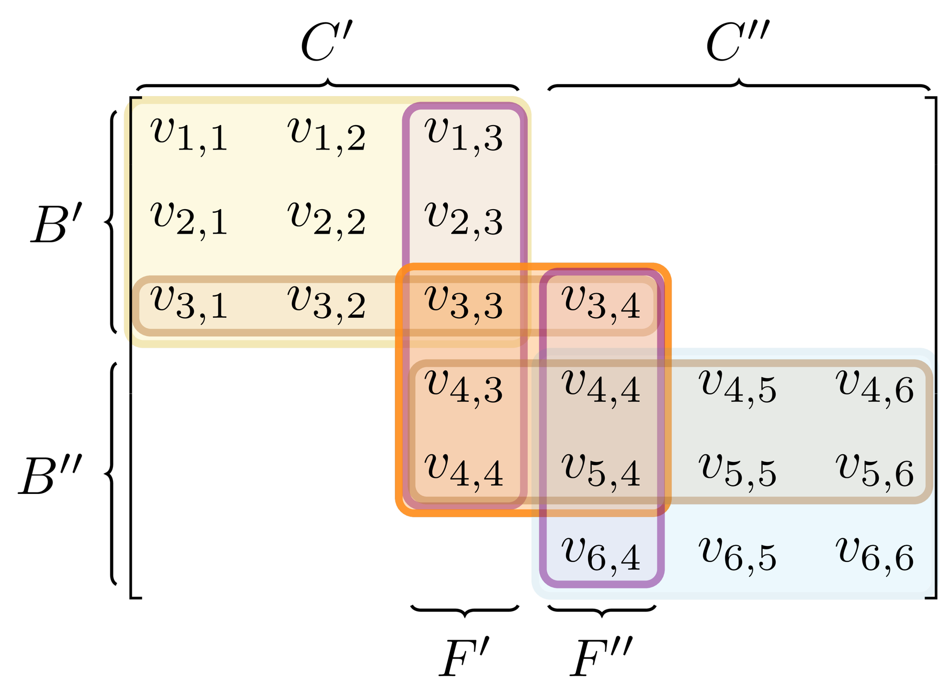

Figure 2: Consider the case where dimensions of B', B'', C', C'' equal 3 and dimensions of E', E'', F', F'' equal 1, 2, 1, 1. Elements of $V_{\mathrm{Seg}} \subset B \otimes C$ lie in subspaces: *Prime* (yellow rectangle), *Bis* (blue rectangle), *VL*, *VR* (both purple), *HR*, *HL* (both brown) and *Mix* (orange)

Idea

We analyse rank one matrices contributing to the minimal decompositions of tensors, and we notice that every element of $V_{\mathrm{Seg}} \subset B \otimes C$ lies in one of the following subspaces of $B \otimes C$:

- $B' \otimes C'$, $B'' \otimes C''$; (*Prime*, *Bis*)
- $E' \otimes (C' \oplus F'')$, $E'' \otimes (F' \oplus C'')$, $(E' \oplus B'') \otimes F''$, $(B' \oplus E'') \otimes F'$; (*VL*, *VR*, *HR*, *HL*)
- $(E' \oplus E'') \otimes (F' \oplus F'')$. (*Mix*).

The core observation is that one can get rid matrices of type Prime and Bis. That is we can always produce a smaller example, which does not have these two types, but if the additivity holds for the smaller one, then it also holds for the original one.

Improved Substitution method

Substitution method by Alexeev-Forbes-Tsimerman [1] says, that if $w \in W' \subset W$, s.t. $R(w) = 1$ then there exists a complementary subspace $\widetilde{W} \subset W$, s. t.

$$\widetilde{W} \oplus \langle w \rangle = W \text{ and } R(\widetilde{W}) = R(W) - 1.$$

We proved, that there exists a choice of a complementary subspace $\widetilde{W}' \subset W' \subset W$, s. t.

$$\widetilde{W}' \oplus \langle w \rangle \oplus W'' = W \text{ and } R(\widetilde{W}) = R(W) - 1.$$

Furthermore, if the rank additivity does not hold for (W', W''), then for $(\widetilde{W}', W'')$ it does not hold as well.

$$\begin{bmatrix} * & * & * & & & & \\ * & * & * & & & & \\ * & * & * & & & & \\ & & & * & * & * & * \\ & & & * & * & * & 0 \\ & & & * & * & 0 & 0 \end{bmatrix} \to \begin{bmatrix} * & * & * & & & & \\ * & * & * & & & & \\ * & * & * & & & & \\ * & * & * & * & * & * & 0 \\ & & & * & * & * & 0 \\ & & & * & * & 0 & 0 \end{bmatrix}$$

Figure 3: After applying the substitution method, rank one matrix (in red) disappears, but residues possibly appear (in blue and red). Rank of the whole tensor drops by one.

$$\begin{bmatrix} * & * & * & & & & \\ * & * & * & & & & \\ * & * & * & & & & \\ & & & * & * & * & * \\ & & & * & * & * & 0 \\ & & & * & * & 0 & 0 \end{bmatrix} \to \begin{bmatrix} * & * & * & & & & \\ * & * & * & & & & \\ * & * & * & & & & \\ & & & * & * & * & 0 \\ & & & * & * & * & 0 \\ & & & * & * & 0 & 0 \end{bmatrix}$$

Figure 4: After applying the **improved** substitution method, blue part possibly changes, but the direct sum structure is preserved. If we had a counterexample for rank additivity property, now we obtained even a smaller one.

References

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