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Secant non-defectivity of Segre-Veronese varieties via collisions of points

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The problem

Classify defective varieties,

i.e., algebraic varieties for which the dimension of some secant variety is smaller than the expected.

In particular, in the case of Segre-Veronese varieties with two factors.

$$\nu_{c,d} : \mathbb{P}\mathbb{C}^{m+1} \times \mathbb{P}\mathbb{C}^{n+1} \longrightarrow \mathbb{P}(\mathrm{Sym}^c\mathbb{C}^{m+1} \otimes \mathrm{Sym}^d\mathbb{C}^{n+1}), \quad ([v],[w]) \mapsto [v^{\otimes c} \otimes w^{\otimes d}]$$

Veronese varieties.

A list of defective cases was known since the beginning of XIX century.

In 1995, Alexander and Hirschowitz proved that the known list of defective cases was complete.

Segre-Veronese varieties with two factors.

In the last 25 years, a list of defective cases has been found by various authors.

(Abrescia, Bocci, Catalisano, Geramita, Gimigliano, Ottaviani,... - see [AB13])

All the known defective cases appear in bi-degrees (c,d) where either $c < 3$ or $d < 3$.

Theorem.

If $c,d \geq 3$, then the Segre-Veronese variety $\nu_{c,d}(\mathbb{P}^m \times \mathbb{P}^n)$ is never defective.

Abo-Brambilla (2013), [AB13] - "the inductive step"

If there are no defective cases in bi-degrees (3,3), (3,4) and (4,4), then there are no defective cases in bi-degrees (c,d) for $c,d \geq 3$.

Galuppi-Oneto (2021), [GO21] - "the base cases"

In bi-degrees (3,3), (3,4) and (4,4) there are no defective cases.

References.

[AB13] H. Abo and M.C. Brambilla, "On the dimensions of secant varieties of Segre-Veronese varieties", Annali di Matematica Pura ed Applicata, 192(1):61–92, 2013.

[GO21] F. Galuppi and A. Oneto, "Secant non-defectivity of via collisions of fat points", arXiv preprint arXiv:2104.02522, 2021.

An interpolation problem.

$$\mathrm{codim}\ \sigma_r(\nu_{c,d}(\mathbb{P}^m \times \mathbb{P}^n)) = \dim I(\mathbb{X})_{c,d}$$

where $\mathbb{X}$ is a scheme of r general 2-fat points in $\mathbb{P}^m \times \mathbb{P}^n$ and $I(\mathbb{X})_{c,d}$ is the part in bidegree (c,d) of its defining ideal.

Upper-bound by degeneration.

$$\exp.\mathrm{codim}\ \sigma_r(\nu_{c,d}(\mathbb{P}^m \times \mathbb{P}^n)) \leq \mathrm{codim}\ \sigma_r(\nu_{c,d}(\mathbb{P}^m \times \mathbb{P}^n)) = \dim I_{c,d}(\mathbb{X}) \leq \dim I_{c,d}(\tilde{\mathbb{X}})$$

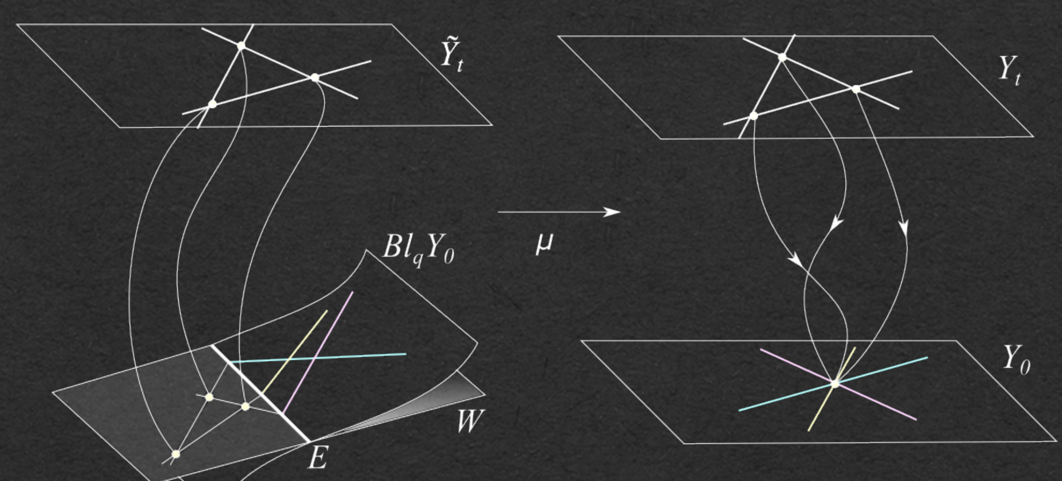
where $\tilde{\mathbb{X}}$ is a degeneration of the scheme $\mathbb{X}$.

Collision of fat points.

Over an N-dimensional variety, the collision of N+1 general 2-fat points is a local scheme such that:

- contains a 3-fat point;
- the restriction on a general line through it has degree 3;
- there are $\binom{N+1}{2}$ lines such that the restriction has degree 4;

namely, a 3-fat point with $\binom{N+1}{2}$ points infinitesimally close.



In our proof [GO21], we consider the degeneration of $\mathbb{X}$ obtained by collapsing m+n+1 among the 2-fat points of $\mathbb{X}$.