Limits of saturated ideals of points

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[a] I was supported by NCN grants 2017/26/E/ST1/00231 and 2019/33/N/ST1/00858

Border rank and secant varieties

Let X be a smooth complex projective **toric variety** with a very ample line bundle L. We consider X as an embedded variety $X \subseteq \mathbb{P}(H^0(X,L)^*)$.

- Let $[F] \in \mathbb{P}(H^0(X,L)^*)$. The *$X$-rank* $\mathrm{r}_X([F])$ of $[F]$ is the least integer r such that $[F] \in \langle p_1, \ldots, p_r\rangle$ for some $p_1, \ldots, p_r \in X$.
- The *r-th secant variety* of X is
$$\sigma_r(X) = \overline{\{[F] \in \mathbb{P}(H^0(X,L)^*) \mid \mathrm{r}_X([F]) \le r\}}.$$
- The *X-border rank* of $[F]$ is the least integer r such that $[F] \in \sigma_r(X)$.

Multigraded Hilbert schemes and the irreducible component $\mathrm{Slip}_{r,X}$

Let $S = S[X] = \bigoplus_{[D]\in \mathrm{Pic}(X)} H^0(X,D)$ be the Cox ring of X. Let $h_{r,X} \colon \mathrm{Pic}(X) \to \mathbb{Z}$ be given by $h_{r,X}([D]) = \min\{\dim_{\mathbb{C}} H^0(X,D),\ r\}$ and let $\mathrm{Hilb}^{h_{r,X}}_{S[X]}$ be the corresponding multigraded Hilbert scheme [4], i.e. the scheme parametrizing homogeneous ideals $I \subseteq S$ such that S/I has Hilbert function $h_{r,X}$.
The closure of the locus of points corresponding to saturated ideals of points is an irreducible component of $\mathrm{Hilb}^{h_{r,X}}_{S[X]}$ denoted $\mathrm{Slip}_{r,X}$ [1].

Border apolarity [1]

Let $\tilde{S} = \bigoplus_{[D]\in \mathrm{Pic}(X)} H^0(X,D)^*$ be the graded dual ring. We have the apolarity action $\lrcorner \colon S \times \tilde{S} \to \tilde{S}$ given by $(\theta \lrcorner f)(\eta) = f(\theta \cdot \eta)$ for $\theta \in S_{[D]}$, $\eta \in S_{[E-D]}$, $f \in \tilde{S}_{[E]}$.
Given $[F] \in \mathbb{P}(H^0(X,L)^*)$ we denote by $\mathrm{Ann}(F)$ the ideal
$$\mathrm{Ann}(F) = \{\theta \mid \theta \lrcorner F = 0\}.$$
We have
$$[F] \in \sigma_r(X) \Leftrightarrow \exists\ [I] \in \mathrm{Slip}_{r,X} \text{ s. t. } I \subseteq \mathrm{Ann}(F).$$

Statement of the problem

Finding criteria (sufficient or necessary) for $[I] \in \mathrm{Hilb}^{h_{r,X}}_{S[X]}$ to be in $\mathrm{Slip}_{r,X}$.

Criterion for $\mathbb{P}^n$ based on smoothness [6]

Let $S = S[\mathbb{P}^n]$ and let $[I] \in \mathrm{Hilb}^{h_{r,\mathbb{P}^n}}_S$ be a closed point corresponding to an ideal that is not saturated. Let d be such that $I_d \neq \overline{I}_d$, where $\overline{I}$ is the saturation of I. Let $J = \overline{I} \cap \mathfrak{m}^d$ and $K = I \cap \mathfrak{m}^d$.

Assume that:

- the natural maps
$$\mathrm{Hom}_S(J, S/J)_0 \to \mathrm{Hom}_S(K, S/J)_0$$
and
$$\mathrm{Hom}_S(K, S/K)_0 \to \mathrm{Hom}_S(K, S/J)_0$$
are surjective;
- $[J] \in \mathrm{Hilb}^h_S$ is a smooth point where h is the Hilbert function of S/J.

Then there is no $[I'] \in \mathrm{Slip}_{r,\mathbb{P}^n}$ such that $I'_{\ge d} = I_{\ge d}$. In particular, $[I] \notin \mathrm{Slip}_{r,\mathbb{P}^n}$.

Ideals defining subschemes contained in line [5][6]

As an application of the above criterion we get the following result concerning ideals defining subschemes contained in line.

Let $[I] \in \mathrm{Hilb}^{h_{r,\mathbb{P}^n}}_{S[\mathbb{P}^n]}$ be a closed point corresponding to an ideal I such that $S[\mathbb{P}^n]/\overline{I}$ has Hilbert function $h_{r,\mathbb{P}^1}$. Then there exists $[I'] \in \mathrm{Slip}_{r,\mathbb{P}^n}$ such that $I_{\ge r-2} = I'_{\ge r-2}$ if and only if $(\overline{I}^2)_{r-2} \subseteq I_{r-2}$.

Example

Let $I = (\alpha_0\alpha_1, \alpha_0\alpha_2, \alpha_0^3, \alpha_1^4) \subseteq S[\mathbb{P}^2]$. Then $[I] \in \mathrm{Hilb}^{h_{4,\mathbb{P}^2}}_{S[\mathbb{P}^2]}$ but $[I] \notin \mathrm{Slip}_{4,\mathbb{P}^2}$ since $\overline{I} = (\alpha_0, \alpha_1^4)$ and $\alpha_0^2 \notin I$.
Therefore, $\mathrm{Hilb}^{h_{4,\mathbb{P}^2}}_{S[\mathbb{P}^2]}$ is reducible.

Quotient construction and lifting maps to Cox rings

Let $\overline{X} = \mathrm{Spec}\, S[X]$ and let $\widehat{X} = \overline{X} \setminus B(\Sigma_X)$ be the complement of the zero set of the irrelevant ideal. Then X is the quotient of $\widehat{X}$ by $\mathrm{Spec}\, \mathbb{C}[\mathrm{Pic}(X)]$ [2]. We denote the quotient map by π_X.
Given a morphism $f \colon X \to Y$ between smooth projective toric varieties there exists a homomorphism of $\mathbb{C}$-algebras $\overline{f}^{\#} \colon S[Y] \to S[X]$ such that

- $\overline{f}^{\#}(S[Y]_{[D]}) \subseteq S[X]_{f^*([D])}$ for every $[D] \in \mathrm{Pic}(X)$;
- $\overline{f}$ restricts to a morphism $\hat{f} \colon \widehat{X} \to \widehat{Y}$;
- $\pi_Y \circ \hat{f} = f \circ \pi_X$ [3].

Criterion based on Cox lift [6]

Let $f \colon X \to Y$ be a morphism between smooth projective toric varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Let r be a positive integer and let $\overline{f}^{\#} \colon S[Y] \to S[X]$ be a lift of f as above. Then

- $\overline{f}^{\#}$ induces a morphism $\pi \colon \mathrm{Hilb}^{h_{r,X}}_{S[X]} \to \mathrm{Hilb}^{h_{r,Y}}_{S[Y]}$ given on closed points by $[I] \mapsto [(\overline{f}^{\#})^{-1}(I)]$;
- The morphism $\pi \colon \mathrm{Hilb}^{h_{r,X}}_{S[X]} \to \mathrm{Hilb}^{h_{r,Y}}_{S[Y]}$ induces a surjection $\mathrm{Slip}_{r,X} \to \mathrm{Slip}_{r,Y}$.

Criterion for $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ [6]

For the product of projective spaces we also have the following necessary condition.

Let $X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ for some positive integers $k \ge 2, n_1, \ldots, n_k$. For $i \in \{1, \ldots, k\}$ let $B_i \subseteq S[X]$ be the extension of the irrelevant ideal of $\mathbb{P}^{n_i}$ under the natural inclusion $S[\mathbb{P}^{n_i}] \to S[X]$. If $[I] \in \mathrm{Slip}_{r,X}$ for some positive integer r, then
$$\dim_{\mathbb{C}} \mathrm{Hom}_{S[X]}\Big(I + (B_i)^2, S[X]/(I + (B_i)^2)\Big)_0$$
is at least $r(n_1 + \ldots + n_k)$ for $i \in \{1, \ldots, k\}$.

Example

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let
$$[I] = [(\alpha_0\alpha_1, \alpha_0\beta_0, \alpha_1\beta_0, \beta_0\beta_1)] \in \mathrm{Hilb}^{h_{2,X}}_{S[X]}.$$
Then $[I] \notin \mathrm{Slip}_{2,X}$ since
$$\dim_{\mathbb{C}} \mathrm{Hom}_{S[X]}(J, S[X]/J)_0 < 4,$$
where $J = I + (\alpha_0, \alpha_1)^2$. In particular, $\mathrm{Hilb}^{h_{2,X}}_{S[X]}$ is reducible.

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