Moduli spaces of quasitrivial rank r sheaves on $\mathbb{P}^3$

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Abstract

A torsion free sheaf E on $\mathbb{P}^3$ is called quasitrivial *if $E^{\vee\vee} = \mathcal{O}_{\mathbb{P}^3}^{\oplus r}$ and $\dim(E^{\vee\vee}/E) = 0$. While such sheaves are always μ-semistable, they may not be Gieseker semistable. We study the moduli spaces of Gieseker semistable quasitrivial sheaves of rank r, denoted by $\mathcal{N}(r, n)$, via the Quot scheme of points $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$, where $n = h^0(E^{\vee\vee}/E)$. Our main result is the construction of an irreducible component of the Gieseker moduli space and, furthermore, we show that this is the only component when $n \leq 10$.*

1. Semistable sheaves with vanishing Chern classes

Recall that a coherent sheaf E on a smooth projective variety X of dimension d is said to be μ-*(semi)stable* with respect to an ample divisor A if E is torsion free and

$$\frac{c_1(F)\cdot A^{d-1}}{\mathrm{rk}(F)} < \ (\leq)\ \frac{c_1(E)\cdot A^{d-1}}{\mathrm{rk}(E)}$$

for every nontrivial subsheaf $F \subset E$. Furthermore, E is *(semi)stable* if

$$\frac{P_F(t)}{\mathrm{rk}(F)} < \ (\leq)\ \frac{P_E(t)}{\mathrm{rk}(E)}$$

where $P_E(t)$ denotes the Hilbert polynomial of the sheaf E with respect to the divisor A.

Next lemma is a key technical result to achieve our results.

Lemma 1.1. Let F be a μ-semistable reflexive sheaf of rank r on $\mathbb{P}^d$ with $d \geq 3$. If $c_1(F) = c_2(F) = 0$, then F is isomorphic to $\mathcal{O}_{\mathbb{P}^d}^{\oplus r}$.

2. Quot scheme and extensions of ideals

Let (φ, Q) be an element of the Quot scheme $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ with $u \in \mathbb{Q}[t]$ such that $\deg(u) \leq d-3$, that is, $\varphi\colon \mathcal{O}_{\mathbb{P}^d}^{\oplus r} \to Q$ is an epimorphism onto a 3-codimensional sheaf Q with Hilbert polynomial $P_Q(t) = u(t)$. Now let $E := \ker\varphi$. In that case, we have a short exact sequence

$$0 \to E \to \mathcal{O}_{\mathbb{P}^d}^{\oplus r} \to Q \to 0. \tag{1}$$

Clearly, E is a torsion free sheaf of rank r with Hilbert polynomial $P_E(t) = r\cdot\binom{t+d}{d} - u(t)$ and Chern classes $c_1(E) = c_2(E) = 0$.

Proposition 2.1. Given $(\varphi, Q) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ with $r > 1$ and $u(t) \in \mathbb{Q}[t]$ such that $\deg(u) \leq d-3$, then the sheaf $E := \ker\varphi$ is strictly μ-semistable.

Next proposition establishes the converse to the previous claim.

Proposition 2.2. Let E be a μ-semistable sheaf of rank r on $\mathbb{P}^d$ with $c_1(E) = c_2(E) = 0$. Then there is $(\varphi, Q) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ with $\deg(u) \leq d-3$ such that $E \cong \ker\varphi$.

Theorem 2.3. If E is a μ-semistable sheaf on $\mathbb{P}^d$ with $\mathrm{rk}(E) \geq 1$ $c_1(E) = c_2(E) = 0$, then E is an extension of ideal sheaves of subschemes of $\mathbb{P}^d$ of codimension at least 3.

Lemma 2.4. Let E be a semistable sheaf on with $c_1(E) = c_2(E) = 0$ on $\mathbb{P}^d$. There are $(\psi, Q_F) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus s}, v(t))$ and $(\psi', Q_G) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus (r-s)}, P_E - v)$ for some polynomial $v \in \mathbb{Q}[t]$ with $\deg(v) \leq d-3$ such that E can be written as an extention

$$0 \to F \to E \to G \to 0,$$

with $F = \ker\psi$ being semistable and $G = \ker\psi'$ being stable.

3. Moduli spaces of semistable sheaves on $\mathbb{P}^d$ with $c_1 = c_2 = 0$

Consider first the following action of $\mathrm{GL}_r \simeq \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r})$ on $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$:

$$g\cdot(\varphi, Q) := (\varphi\circ g, Q). \tag{2}$$

Propositions 2.1 and 2.2 lead to the following theorem characterizing the set of isomorphism classes of μ-semistable sheaves.

Theorem 3.1. There is a bijection between the set of isomorphism classes of μ-semistable sheaves E of rank r with Chern classes $c_1(E) = c_2(E) = 0$ on $\mathbb{P}^d$, and the set of orbits $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)/\mathrm{GL}_r$ for $u \in \mathbb{Q}[t]$ given by $u(t) := r\cdot\binom{t+d}{d} - P_E(t)$

Next theorem provides a characterization of those $(\varphi, Q) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, u)$ for which $\ker\varphi$ is semistable.

Theorem 3.2. Let (φ, Q) in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ for $u \in \mathbb{Q}[t]$ with $\deg(u) \leq d-3$, and let $E = \ker\varphi$. Then E is not $(semi)stable$ if, and only if, there is a torsion free sheaf $F \hookrightarrow E$ with $F = \ker\psi$ for some (ψ, Q_F) in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus s}, v)$ satisfying $0 < s < r$ and

$$v < \ (\leq)\frac{s\cdot u}{r}.$$

Let (φ, Q) be an element in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ and let $E = \ker\varphi$. We can also relate the (semi)stability of E given in the above theorem with GIT-stability applied to the GL_r-action on $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ given by (2).

Theorem 3.3. Let (φ, Q) be an element in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ and let $E = \ker\varphi$. Then (φ, Q) is GIT-(semi)stable with respect to the GL_r action in display (2) if, and only if, E is (semi)stable.

4. Moduli spaces of semistable quasitrivial sheaves on $\mathbb{P}^3$

The first observation is that $\mathcal{N}(1, n)$ coincides with the Hilbert scheme of 0-dimensional subschemes of $\mathbb{P}^3$ with length n. We will therefore focus on $r \geq 2$, and we were able to prove the following results.

Proposition 4.1. $\mathcal{N}(r, n) = \emptyset$ for $r > n$.

Proposition 4.2. If $n > 1$, then $\mathcal{N}^{\mathrm{st}}(n, n) = \emptyset$, and $\mathcal{N}(n, n) = \mathrm{Sym}^n(\mathbb{P}^3)$.

Using the results from the previous sections applied to $\mathbb{P}^3$ we have that $\mathcal{N}(r, n)$ is the GIT–quotient of $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$ by GL_r, and since $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$ is irreducible for $n \leq 10$, we get the following theorem.

Theorem 4.3. $\mathcal{N}(r, n)$ is irreducible for $n \leq 10$.

5. Irreducible component of $\mathcal{N}(r, n)$

Theorem 5.1. Let (φ, Q) be an element in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$ with Q the structural scheme of a closed subscheme of $\mathbb{P}^3$ which is in the smooth component of the Hilbert scheme of n points in $\mathbb{P}^3$. Let $E = \ker\varphi$ such that E is stable. Then $\mathrm{ext}^1(E, E) = 2n + rn - r^2 + 1$.

Now let $\mathcal{H}^i$ the universal sheaf of $\mathrm{Hilb}^i(\mathbb{P}^3)$ on $\mathbb{P}^3 \times \mathrm{Hilb}^i$. Consider the following diagram

$$\begin{array}{ccc} & \mathrm{Hilb}^1 \times \mathrm{Hilb}^{n-1} & \\ & \uparrow f & \\ & \mathbb{P}^3 \times \mathrm{Hilb}^1 \times \mathrm{Hilb}^{n-1} & \\ \swarrow p_1 & & p_2 \searrow \\ \mathbb{P}^3 \times \mathrm{Hilb}^1 & & \mathbb{P}^3 \times \mathrm{Hilb}^{n-1}, \end{array}$$

Let $Y := \{(y, Z) \in \mathrm{Hilb}^1 \times \mathrm{Hilb}^{n-1} \mid y \notin Z\} \overset{\text{open}}{\subset} \mathrm{Hilb}^1 \times \mathrm{Hilb}^{n-1}$, $X := \mathbb{P}^3 \times Y$, and $\pi : X \to Y$ be the projection. Define $\mathcal{E}^i := \mathcal{E}xt^i_\pi(p_1^*\mathcal{H}^1, p_2^*\mathcal{H}^{n-1})$. Let (y, Z) be a point in Y and let $\tau^i(y, Z)$ be the base change morphism:

$$\tau^i(y, Z) : \mathcal{E}xt^i_\pi(p_1^*\mathcal{H}^1, p_2^*\mathcal{H}^{n-1}) \otimes k(y, Z) \to \mathrm{Ext}^i_{\mathbb{P}^3}(I_y, I_Z).$$

We can show that $\mathcal{E}^1$ is locally free on Y and that there is an universal extension $\mathcal{H}$ on $X \times H$ with $H = \mathbb{P}((\mathcal{E}^1)^*)$ such that for every $h \in H$, the restriction $\mathcal{H}|_h$ is a nonsplit extension of two sheaf of ideals of 0-dimensional subschemes of $\mathbb{P}^3$ of the following form

$$0 \to I_y \to E \to I_Z \to 0,$$

moreover, the dimension of this family is equal to $4n - 3$.

Now $\mathcal{N}(2, n)$ is a coarse moduli space, so our family $\mathcal{H}$ on $X \times H$ gives us a morphism $H \to \mathcal{N}(2, n)$. Taking the closure of the image H in $\mathcal{N}(2, n)$, denoting by $\overline{H}$, by Theorem 5.1 we get the following theorem.

Theorem 5.2. Let $n > 2$. Then $\overline{H}$ is an irreducible component of $\mathcal{N}(2, n)$ of dimension $4n - 3$.

To construct an irreducible component for $\mathcal{N}(r, n)$ for $r > 2$, we use a similar idea and the case $r = 2$ as an induction step.

Theorem 5.3. Let $r < n$ be positive integers. There is an irreducible component of $\mathcal{N}(r, n)$ given by $\overline{H}$ of dimension $2n + rn - r^2 + 1$.

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