Graph states and the Variety of Principal Minors

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Algebraic objects

Let $X, Y, Z \in U(2, \mathbb{C})$ be the elementary Pauli matrices.
The *n-fold Pauli group* is $\mathcal{P}_n = \left\{ \pm i^\delta Z^{\mu_1} X^{\nu_1} \otimes \ldots \otimes Z^{\mu_n} X^{\nu_n} \mid \delta, \mu_i, \nu_i \in \{0, 1\} \right\}$ in $\mathrm{U}(2^n, \mathbb{C})$.

$$\mathcal{S}(\mathcal{P}_n) = \{ \langle M_1, \ldots, M_n \rangle \mid M_i \in \mathcal{P}_n \text{ independent and mutually commuting} \}.$$

A n-qubit *stabilizer state* $|\phi\rangle$ is the common eigenvector with eigenvalue $+1$ of a certain $S \in \mathcal{S}(\mathcal{P}_n)$.
If S is defined by a graph, $|\phi\rangle$ is said *graph state*.

Local Clifford action

The *n-fold local Clifford group* is $\mathcal{C}_n^{\mathrm{loc}} = \left\{ U_1 \otimes \ldots \otimes U_n \mid U_i \in N_{U(2,\mathbb{C})}(\langle iX, iY, iZ \rangle) \right\}$.

$$\begin{array}{ccc} \mathcal{C}_n^{\mathrm{loc}} \times \mathcal{P}_n & \longrightarrow & \mathcal{P}_n \\ (U_1 \otimes \ldots \otimes U_n, A_1 \otimes \ldots \otimes A_n) & \mapsto & (U_1 A_1 U_1^H) \otimes \ldots \otimes (U_n A_n U_n^H) \end{array}.$$

This action translates into actions on $\mathcal{S}(\mathcal{P}_n)$ and on the set of stabilizer states. In particular, graph states are representatives for stabilizer states for $\mathcal{C}_n^{\mathrm{loc}}$ action **[1]**.

Geometric objects

Fix J a symplectic form over $\mathbb{F}_2^{2n}$. Consider the set of maximal (fully) isotropic subspaces of $\mathbb{P}_2^{2n-1}$

$$\mathcal{I}^n = \left\{ W \subset \mathbb{P}_2^{2n-1} \mid \forall P, Q \in W,\ \langle P, Q \rangle_J = 0,\ \dim W = n - 1 \right\} \overset{\text{Plücker}}{\simeq} \mathrm{LG}_{\mathbb{F}_2}(n, 2n).$$

$\mathrm{LG}_{\mathbb{F}_2}(n, 2n)$ is isomorphic to $\mathcal{Z}_n \subset \mathbb{P}_2^{2^n-1}$, the *variety of principal minors* of $n \times n$ symmetric matrices.

Lagrangian mapping

$$\begin{array}{ccccccc} \mathcal{S}(\mathcal{P}_n) & \dashrightarrow & \mathcal{I}^n & \xrightarrow{\ \mathrm{PI}\ } & \mathrm{LG}_{\mathbb{F}_2}(n, 2n) & \xrightarrow{\ \pi \simeq\ } & \mathcal{Z}_n \\ \langle M_1, \ldots, M_n \rangle & \dashrightarrow & H_{P_1,\ldots,P_n} & & [1 : a_{ij} : A_{\{i,j\}\{s,t\}} : \ldots] & \xmapsto{\ \pi\ } & [1 : a_{ii} : A_{[i,j]} : \ldots] \end{array}$$

$$= \qquad\qquad \uparrow$$

$$\left\langle \begin{bmatrix} e_1 \\ \hline a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}, \ldots, \begin{bmatrix} e_n \\ \hline a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix} \right\rangle \xmapsto{\ \mathrm{PI}\ } \begin{bmatrix} e_1 \\ \hline a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} \wedge \ldots \wedge \begin{bmatrix} e_n \\ \hline a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix}$$

Local symplectic action

The *local symplectic group* is $\mathrm{Sp}_{2n}^{\mathrm{loc}}(\mathbb{F}_2) = \left\{ \begin{bmatrix} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_n & & & b_n \\ c_1 & & & d_1 & & \\ & \ddots & & & \ddots & \\ & & c_n & & & d_n \end{bmatrix} \in \mathrm{Sp}(2n, \mathbb{F}_2) \;\middle|\; a_i d_i - b_i c_i \neq 0 \right\}$.

The action $\mathcal{C}_n^{\mathrm{loc}} \curvearrowright \mathcal{S}(\mathcal{P}_n)$ corresponds to the action $\begin{array}{ccc} \mathrm{Sp}_{2n}^{\mathrm{loc}}(\mathbb{F}_2) \times \mathcal{I}^n & \longrightarrow & \mathcal{I}^n \\ (U, H_{P_1,\ldots,P_n}) & \mapsto & H_{UP_1,\ldots,UP_n} \end{array}$.

The action $\mathrm{Sp}_{2n}^{\mathrm{loc}}(\mathbb{F}_2) \curvearrowright \mathcal{Z}_n$ is equivalent to an action $\mathrm{SL}(2, \mathbb{F}_2)^{\times n} \curvearrowright \mathcal{Z}_n$ **[3]**.
Moreover, there is an action of $\mathrm{SL}(2, \mathbb{F}_2)^{\times n} \rtimes \mathfrak{S}_n$ on $\mathcal{Z}_n$ which is maximal **[4]**.

Theorem (G. - Holweck 2021, [2])

The Lagrangian mapping induces a bijection between the $\mathcal{C}_n^{\mathrm{loc}} \rtimes \mathfrak{S}_n$-orbits of $\mathcal{S}(\mathcal{P}_n)$ and the $\mathrm{SL}(2, \mathbb{F}_2)^{\times n} \rtimes \mathfrak{S}_n$-orbits of $\mathcal{Z}_n$. In particular, there is a **1-1** correspondence between representatives of the graph states for the $\mathcal{C}_n^{\mathrm{loc}} \rtimes \mathfrak{S}_n$ action, and the normal forms of the $\mathrm{SL}(2, \mathbb{F}_2)^{\times n} \rtimes \mathfrak{S}_n$-orbits of $\mathcal{Z}_n$.

Happy Birthday, Giorgio!

[1] Dehaene, De Moor, Van den Nest - "Graphical description of the action of local Clifford transformations on graph states", Physical Rev. A 69.2 (2004), p. 022316
[2] Galgano, Holweck - "Graph states and the Variety of Principal Minors", 2021, *preprint*
[3] Holtz, Sturmfels - "Hyperdeterminantal relations among symmetric principal minors", Journal of Algebra 316.2 (2007), pp. 634-648.
[4] Oeding - "*G*-varieties and the principal minors of symmetric matrices", Texas A&M University, 2009

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