Minimal bandwidth $\mathbb{C}^*$-actions on generalized Grassmannians

Alberto Franceschini

Università degli Studi di Trento

Introduction

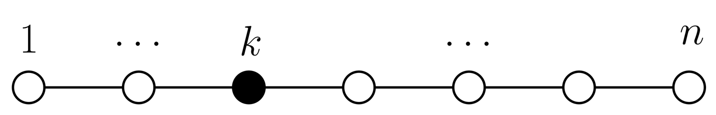
Studying projective varieties through the actions of tori is an idea that goes back to [4]. In [2] the authors propose an algebraic counterpart for the theory, emphasizing the case of rank one torus actions. We test the theory on a big class of examples, generalized Grassmannians, trying to use them as cornerstone of a wider class of examples.

Definitions

Grassmannians:

$$\mathbb{G}(\mathbb{P}^k,\mathbb{P}^n) := \{\mathbb{P}^k : \mathbb{P}^k \subset \mathbb{P}^n\} \subset \mathbb{P}(\wedge^{k+1}\mathbb{C}^{n+1})$$

We can represent it as a Dynkin diagram of type A_n marked in the k-th node (following [1] for the numbering):



$\mathrm{A}_n(k) = \mathbb{G}(\mathbb{P}^k,\mathbb{P}^n)$

Alternatively, one can prove that

$$\mathbb{G}(\mathbb{P}^k,\mathbb{P}^n) = G/P_k$$

where P_k is a maximal parabolic subgroup for $G = \mathrm{Sl}_{n+1}$.

Generalized Grassmannians: Let G be a semisimple group whose Lie algebra $\mathfrak{g}$ is associated to the Dynkin diagram $\mathcal{D}$ and let P_k be a maximal parabolic subgroup corresponding to the simple root α_k. Then

$$\mathcal{D}(k) := G/P_k.$$

Torus actions: Let X be a smooth projective variety with $L \in \mathrm{Pic}(X)$ ample and an action of $H \simeq (\mathbb{C}^*)^n$. We denote

$$M(H) := \mathrm{Hom}(H, \mathbb{C}^*) \simeq \mathbb{Z}^n.$$

The fixed-point locus decomposes as

$$X^H = Y_1 \sqcup \ldots \sqcup Y_s \quad \text{and} \quad \mathcal{Y} = \{Y_1, \ldots, Y_s\}.$$

A *weight map* is

$$\mu_L : \mathcal{Y} \to M(H)$$
$$Y \mapsto \text{the } H\text{-weight of the fiber } L|_Y \to Y.$$

Polytope of fixed points:

$$\Delta(X) := \mathrm{ConvexHull}(\mu_L(Y) : Y \in \mathcal{Y}) \subset M(H) \otimes \mathbb{R}.$$

Consider a maximal torus $H \subset P_k \subset G$, then [5]

$$\Delta(\mathcal{D}(k)) \simeq \mathrm{ConvexHull}(w(\omega_k) : w \in W)$$

where ω_k is the k-th fundamental weight and W is the Weyl group associated to the couple (G, H).

Downgrading: Let $H' \subset H$ be a sub-torus acting on X, then we have

$$\iota^* : M(H) \to M(H')$$

and we have again $\mu'_L : \mathcal{Y}' \to M(H')$ where $\mathcal{Y}'$ is the new set of fixed-point components. We are interested in the case $H' \simeq \mathbb{C}^*$.

Source & sink: Unique fixed-point components Y_+, Y_- such that for a generic $x \in X$ we have

$$\lim_{t\to 0} t \cdot x \in Y_+ \quad \text{and} \quad \lim_{t\to\infty} t \cdot x \in Y_-.$$

Alternatively [2], Y is a

- source if $\Delta(X) \subset \mu'_L(Y) + \mathbb{R}_{\leq 0}$,
- sink if $\Delta(X) \subset \mu'_L(Y) + \mathbb{R}_{\geq 0}$.

Bandwidth:

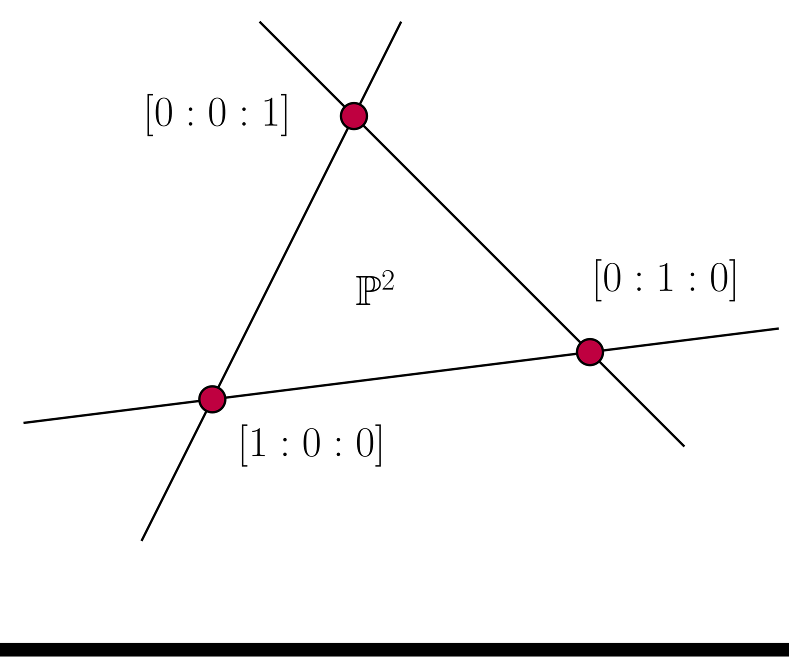
$$|\mu'_L| := \mu'_L(Y_+) - \mu'_L(Y_-).$$

Example: torus actions on $\mathbb{P}^2$

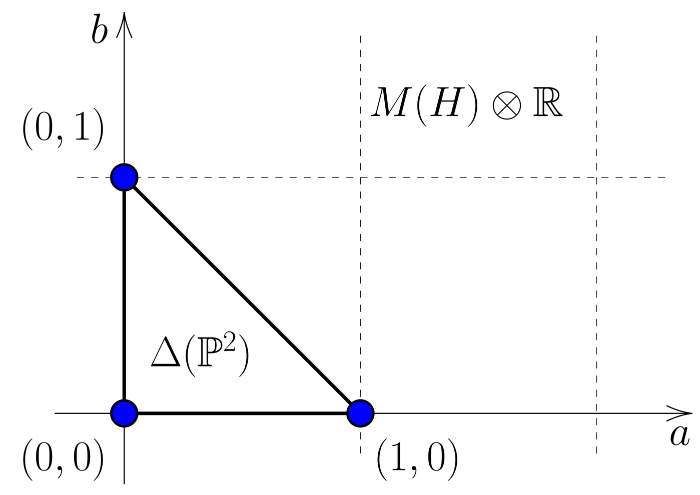
Let us start with the standard action of $H \simeq (\mathbb{C}^*)^2$ on $\mathbb{P}^2 = \mathrm{A}_2(1)$:

$$(s,t) \cdot [x_0 : x_1 : x_2] \mapsto [x_0 : sx_1 : tx_2].$$

There are three fixed points:



Choose $L = \mathcal{O}(1)$, then we have a weight map $\mu := \mu_{\mathcal{O}(1)}$ that sends fixed points to fixed points:



A downgrading μ' for a $\mathbb{C}^*$-action is given, for example, by the choice $t = s$, then

$$s \cdot [x_0 : x_1 : x_2] \mapsto [x_0 : sx_1 : sx_2],$$

whose fixed-point components are

$$Y_0 = [1 : 0 : 0] \qquad \text{with } \mu'(Y_0) = 0,$$
$$Y_1 = \{x_1 - x_2 = 0\} \simeq \mathbb{P}^1 \qquad \text{with } \mu'(Y_1) = 1,$$

hence we obtain a bandwidth one $\mathbb{C}^*$-action on $\mathbb{P}^2$.

Another choice could be μ'', given by $t = s^2$,

$$s \cdot [x_0 : x_1 : x_2] \mapsto [x_0 : sx_1 : s^2x_2]$$

and this time the fixed-point components are

$$Y_0 = [1 : 0 : 0] \qquad \text{with } \mu''(Y_0) = 0,$$
$$Y_1 = [0 : 1 : 0] \qquad \text{with } \mu''(Y_1) = 1,$$
$$Y_2 = [0 : 0 : 1] \qquad \text{with } \mu''(Y_2) = 2,$$

so we obtain a bandwidth two $\mathbb{C}^*$-action on $\mathbb{P}^2$.

Note that these actions can be obtained by projecting $\Delta(\mathbb{P}^2)$ on the lines $b = a$ and $b = 2a$.

Fundamental $\mathbb{C}^*$-actions

Again, $X = \mathcal{D}(k)$ with the action of the maximal torus H. Consider a weight $\omega = \Sigma_{i=1}^n q_i(\omega)\omega_i \in M(H)$, then

$$\omega_i^\vee : M(H) \to \mathbb{Z}$$
$$\omega \mapsto \lfloor q_i(\omega) \rfloor.$$

A *fundamental* $\mathbb{C}^*$*-action* is given by $H_i \simeq \mathbb{C}^*$ such that the downgrading $H_i \subset H$ is given by $\omega_i^\vee$.

Lemma 1 [3]: Let $H' \subset H$ be any 1-dimensional sub-torus acting on X with weight map $\mu' : \mathcal{Y}' \to \mathbb{Z}$. Then there exists a fundamental $\mathbb{C}^*$-action H_j with weight map $\mu_j = \omega_j^\vee \circ \mu : \mathcal{Y} \to \mathbb{Z}$ such that

$$|\mu_j| \leq |\mu'|.$$

Lemma 2 [3]: Consider the fixed points for the H-action given by the identity $e \in W$ and by the longest element $w_\circ \in W$. Then, for any fundamental $\mathbb{C}^*$-action H_j on X,

$$\mu_j(Y_+) = \omega_j^\vee(\omega_k),$$
$$\mu_j(Y_-) = \omega_j^\vee(w_\circ\omega_k).$$

Theorem [3]:

$$|\mu_j| = \omega_j^\vee(\omega_k) - \omega_j^\vee(w_\circ\omega_k)$$

and the $\mathbb{C}^*$-action of minimal bandwidth for X is given by a fundamental one.

Transversal group $G^\perp$

Consider a fundamental $\mathbb{C}^*$-action H_j on the Lie algebra $\mathfrak{g}$, this gives us a $\mathbb{Z}$-grading given by

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \left(\oplus_{\alpha\in\Phi\,:\,\omega_j^\vee(\alpha)=0}\ \mathfrak{g}_\alpha\right)$$
$$\mathfrak{g}_m = \oplus_{\alpha\in\Phi\,:\,\omega_j^\vee(\alpha)=m}\ \mathfrak{g}_\alpha$$

where Φ is the root system associated to the couple (G, H). One can prove that $\oplus_{m\geq 0}\,\mathfrak{g}_m = \mathfrak{p}_j$, where $\mathfrak{p}_j$ is the Lie algebra for the maximal parabolic subgroup P_j.

Proposition [8]: There exists $G_0 \subset G$ reductive such that $\mathrm{Lie}(G_0) = \mathfrak{g}_0$. Moreover, G_0 is the Levi part of P_j.

Then one can write $\mathfrak{g}_0 = \mathfrak{g}^\perp \oplus \mathfrak{a}$ with $\mathfrak{g}^\perp$ a semisimple Lie algebra and $\mathfrak{a} = \langle \alpha_j^\vee \rangle$.

Transversal group: $G^\perp \subset G$ such that $\mathrm{Lie}(G^\perp) = \mathfrak{g}^\perp$.

Explicitly, $G^\perp$ has an associated Dynkin diagram $\mathcal{D}^\perp$ obtained by $\mathcal{D}$ removing the j-th node.

Consider the embedding $X \subset \mathbb{P}(V)$. Then one have a $\mathbb{Z}$-grading on V given by the fundamental $\mathbb{C}^*$-action H_j:

$$V = \oplus_{\lambda\in M(H)} V_\lambda = \oplus_{m\in\mathbb{Z}} \left(\oplus_{\omega_j^\vee(\lambda)=m} V_\lambda\right).$$

Then

$$X^{H_j} = \sqcup_{m\in\mathbb{Z}} \left(X \cap \mathbb{P}(V_m)\right).$$

Theorem [3]: $X \cap \mathbb{P}(V_m)$ is a finite union of disjoint $G^\perp$-orbits.

Bandwidth one: smooth drums

In [6] the authors prove the following result.

Theorem: A smooth projective variety X with $\rho_X = 1$ admits a bandwidth one $\mathbb{C}^*$-action if and only if X is a horospherical variety listed in [7].

Corollary [3]: A generalized Grassmannian with a bandwidth one $\mathbb{C}^*$-action is a projective space, a Grassmannian, an even quadric hypersurface or a spinor variety:

Variety X	Fundamental $\mathbb{C}^*$-action H_j
$\mathbb{P}^{2n-1} = \mathrm{A}_{2n-1}(1) \simeq \mathrm{C}_n(1)$	H_1
$\mathbb{G}(\mathbb{P}^k,\mathbb{P}^n) = \mathrm{A}_n(k)$	H_1 or H_n
$\mathcal{Q}^{2n-2} = \mathrm{D}_n(1)$	H_{n-1} or H_n
$\mathcal{S}_{n-1} = \mathrm{D}_n(n) \simeq \mathrm{B}_{n-1}(n-1)$	H_1

One can show that also the quasi-homogeneous varieties admits a transversal group $G^\perp$.

Bandwidth two: open problems

Corollary [3, 6]: A generalized Grassmannian with a bandwith two $\mathbb{C}^*$-action is one of the following:

$$X = \begin{cases} \text{any generalized Grassmannian of type } \mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n \\ \text{the Cartan variety } \mathrm{E}_6(1) \simeq \mathrm{E}_6(6) \\ \text{adjoint varieties } \mathrm{E}_6(2), \mathrm{E}_7(1) \\ \text{Legendrian variety } \mathrm{E}_7(7) \\ \text{smooth hyperplane section of Cartan variety } \mathrm{F}_4(4) \\ \text{odd quadric hypersurface } \mathrm{G}_2(1) \simeq \mathrm{B}_3(1) \end{cases}.$$

Problem 1: Find an example of smooth projective non-homogeneous variety with a bandwidth two $\mathbb{C}^*$-action and a transversal group $G^\perp$.

Problem 2: Characterize smooth projective varieties admitting a bandwidth two $\mathbb{C}^*$-action.

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