Dimension of Tensor Network Varieties

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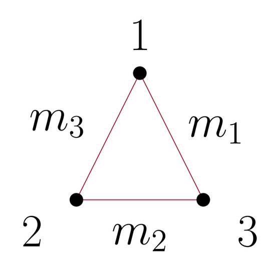
Graph Tensor

Let $\Gamma = (\mathbf{v}(\Gamma), \mathbf{e}(\Gamma))$ be a graph, without loops and multiple edges.

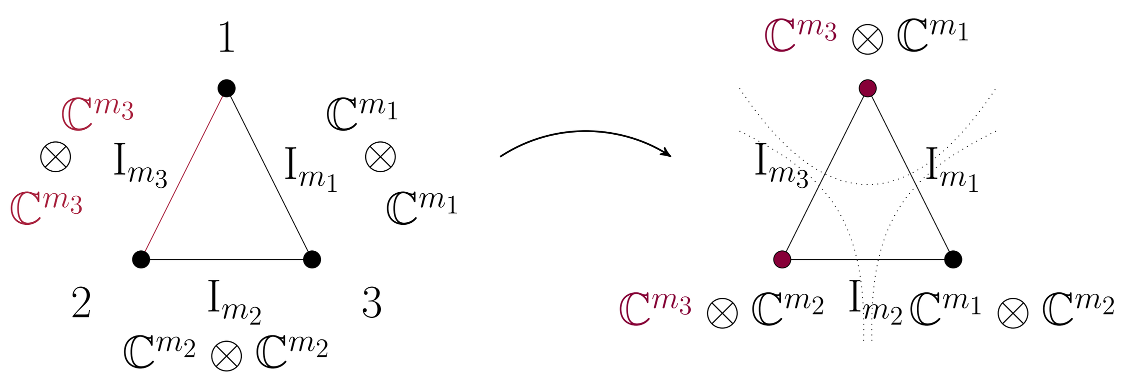
- Bond dimensions: a set of weights

$$\mathbf{m} = (m_e : e \in \mathbf{e}(\Gamma))$$

associated to the edges of the graph.



- For every edge $e \in \mathbf{e}(\Gamma)$, take $\mathrm{I}_{m_e} \in \mathbb{C}^{m_e} \otimes \mathbb{C}^{m_e}$



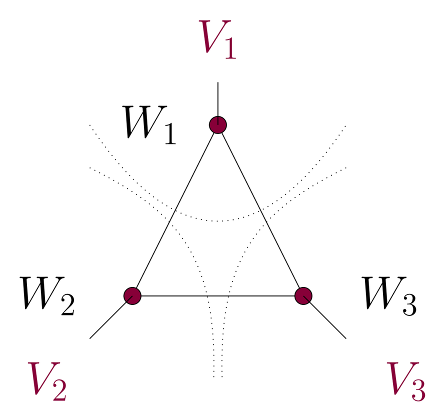
Definition 1 (Graph Tensor) *Define* $W_i = \bigotimes_{e \ni i} \mathbb{C}^{m_e}$.

$$T(\Gamma, \mathbf{m}) = \bigotimes_{e \in \mathbf{e}(\Gamma)} \mathrm{I}_{m_e} \in \bigotimes_{e \in \mathbf{e}(\Gamma)} \mathbb{C}^{m_e} \otimes \mathbb{C}^{m_e} \simeq \bigotimes_{v \in \mathbf{v}(\Gamma)} W_v.$$

- Local dimensions: a set of weights

$$\mathbf{n} = (n_v : v \in \mathbf{v}(\Gamma))$$

associated to the vertices of the graph.
- Define $V_v = \mathbb{C}^{n_v}$, for $v = 1, \dots, d$.



Tensor Network Variety

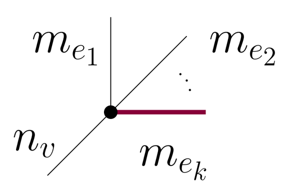
Definition 2 *The tensor network variety in* $V = V_1 \otimes \cdots \otimes V_d$ *associated to* $(\Gamma, \mathbf{m}, \mathbf{n})$ *is*

$$\mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} = \overline{\left\{T \in V : T = (X_1 \otimes \cdots \otimes X_d) \cdot T(\Gamma, \mathbf{m}), X_j \in \mathrm{Hom}(W_j, V_j)\right\}}.$$

Reduction of parameters: fix $v \in \mathbf{v}(\Gamma)$ and let $\mathbf{e}(v) = \{e_1, \dots, e_k\}$.
Assume $m_{e_1} \leq \cdots \leq m_{e_k}$. If $\quad m_{e_k} > n_v \cdot m_{e_1} \cdots m_{e_{k-1}}, \quad m_{e_k}$ is overabundant

then $\quad \mathcal{TNS}_{\mathbf{m},\mathbf{n}} = \mathcal{TNS}_{\overline{\mathbf{m}},\mathbf{n}}$

where $\overline{\mathbf{m}}$ is defined by $\overline{m}_e = m_e$ if $e \neq e_k$ and $\overline{m}_{e_k} = n_v \cdot m_1 \cdots m_{e_{k-1}}$.



Study the Dimension

Theorem 3 *Let* $(\Gamma, \mathbf{m}, \mathbf{n})$ *be a subcritical tensor network with no overabundant bond dimension. Then*

$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} \leq \left[\textstyle\sum_{v \in \mathbf{v}(\Gamma)} N_v n_v - d + 1\right] - \textstyle\sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1) + \dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X)$$

where $N_v = \prod_{e \ni v} m_e$ *and* $X = X_1 \otimes \cdots \otimes X_d$ *with* $X_v \in \mathrm{Hom}(W_v, V_v)$ *generic.*

Theorem 4 *Let* $(\Gamma, \mathbf{m}, \mathbf{n})$ *be a supercritical tensor network. Write* $N_v = \prod_{e \ni v} m_e$. *Then*

$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} = \sum_{v \in \mathbf{v}(\Gamma)} n_v N_v - d + 1 - \sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1).$$

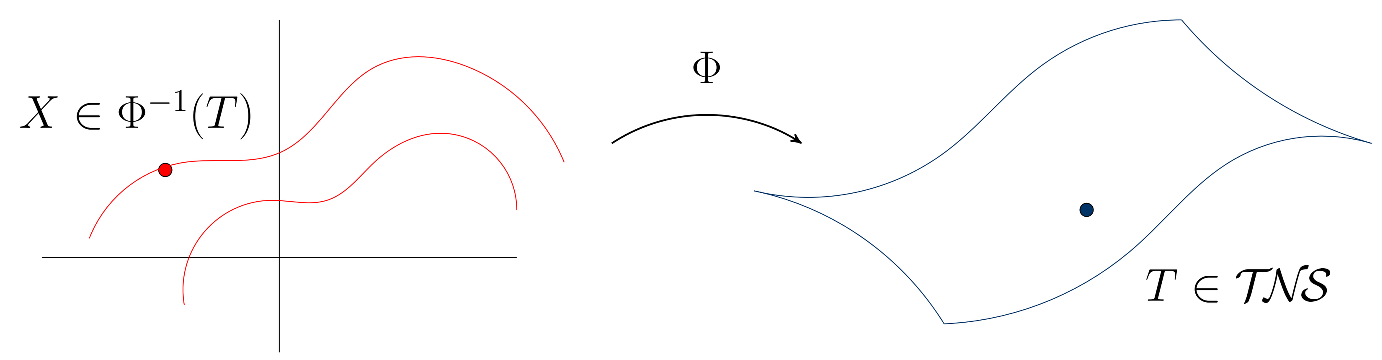
Parametrisation & Theorem of the Fiber

The parametrisation of a dense open subset of $\mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$ is given by

$$\Phi : \mathrm{Hom}(W_1, \dots, W_d; V_1, \dots, V_d) \to V_1 \otimes \cdots \otimes V_d$$
$$(X_1 \otimes \cdots \otimes X_d) \mapsto (X_1 \otimes \cdots \otimes X_d) \cdot T(\Gamma, \mathbf{m})$$

The Theorem of Dimension of the Fibers provides

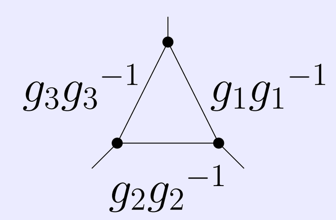
$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} = \dim \left[\mathrm{Hom}(W_1, \dots, W_d; V_1, \dots, V_d)\right] - \dim \Phi^{-1}(T)$$



Consider $X \in \Phi^{-1}(T)$. Then the fiber contains the $\mathcal{G}_{\Gamma,\mathbf{m}}$-orbit of X, where

Definition 5 (Gauge Subgroup)

$$\mathcal{G}_{\Gamma,\mathbf{m}} \simeq \bigtimes_{e \in \mathbf{e}(\Gamma)} PGL_{m_e}.$$



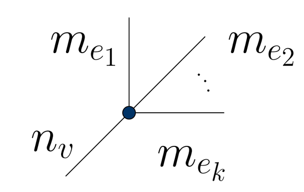
$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} \leq \dim \mathrm{Hom}(W_1, \dots, W_d; V_1, \dots, V_d) - \dim(\mathcal{G}_{\Gamma,\mathbf{m}} \cdot X)$$

$$= \underbrace{\left(\sum_{v \in \mathbf{v}(\Gamma)} N_v n_v - d + 1\right)}_{\dim \mathrm{Hom}(W_1, \dots, W_d; V_1, \dots, V_d)} - \underbrace{\sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1)}_{\dim \mathcal{G}_{\Gamma,\mathbf{m}}} + \dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X)$$

SubSuperCritical Vertex

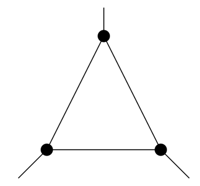
A vertex $v \in \mathbf{v}(\Gamma)$ is called

- subcritical if $\dim W_v = \prod_{e \ni v} m_e \geq n_v$,
- supercritical if $\dim W_v = \prod_{e \ni v} m_e \leq n_v$,
- critical if v is both subcritical and supercritical.



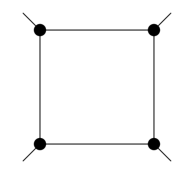
Defective Cases

$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} < \left(\sum_{v \in \mathbf{v}(\Gamma)} N_v n_v - d + 1\right) - \sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1) + 0.$$



$\mathbf{m} = (2, 2, 2)$

$\mathbf{n}$	lower bound	upper bound
$(2,2,2)$	8	8
$(2,2,3)$	12	12
$(2,2,4)$	16	16
$(2,3,3)$	18	18
* $(2,3,4)$	22	24
* $(2,4,4)$	26	29
$(3,3,3)$	25	25
$(3,3,4)$	29	29
$(3,4,4)$	31	31
$(4,4,4)$	37	37



$\mathbf{m} = (2, 2, 2, 2)$

$\mathbf{n}$	lower bound	upper bound
* $(2,2,2,2)$	15	16
* $(2,2,2,3)$	20	21
* $(2,2,2,4)$	24	25
* $(2,3,2,3)$	24	25
* $(2,3,2,4)$	28	29
* $(2,4,2,4)$	32	33

Theorem 6 *If* $\mathbf{m} = (2, 2, 2), \mathbf{n} = (2, n_2, 4)$ *then*

$$\mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}} = \overline{\left\{T \in \mathbb{C}^2 \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^4 : T(\mathbb{C}^{2*}) \cap \sigma_2 \text{ contains at least two points}\right\}}.$$

Theorem 7 *If* $\mathbf{m} = (2, 2, 2, 2), \mathbf{n} = (2, 2, 2, 2)$ *then* $\mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}}$ *is a hypersurface of degree* 6.

Conjecture

Conjecture 8 *Let* $d \geq 3$, $\mathbf{m} = (2, \dots, 2)$ *and* $\mathbf{n} = (n_1, \dots, n_d)$ *with* $n_j \geq 2$. *Then* $\mathcal{TNS}^{C_d}_{\mathbf{m},\mathbf{n}}$ *has the expected dimension, except for:*

- *if* $d = 3$: $\mathbf{n} = (2, n_2, n_3)$, *with* $n_2 \geq 3$, $n_3 \geq 4$ *and cyclic permutations,*
- *if* $d = 4$: $\mathbf{n} = (2, n_2, 2, n_4)$ *with* $n_2, n_4 \geq 2$ *and cyclic permutations.*

References

[1] H. Derksen and V. Makam. *Maximum likelihood estimation for matrix normal models via quiver representations.* arXiv:2007.10206, 2020.

[2] A. Bernardi, C. De Lazzari, F. Gesmundo, *Dimension of Tensor Network varieties.* Preprint on arXiv, Macaulay2 code.