Analytical classification of triple points
Hilbert Functions of aCM Veronese subvarieties

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Newton-Puiseux algorithm and its application to triple points

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Question

Is there a classification up to analytical equivalence of the reduced algebraic plane curves singularities?

ADE Singularities

A partial answer to this question is given by the so called ADE singularities. Let

$$\mathcal{C} : f = 0$$

be a reduced algebraic plane curve and $P \in \mathcal{C}$ be a point such that $m_P(\mathcal{C}) = 2, 3$. Then, P is analitycally equivalent to the singularity that a certain 'representative' curve has in $O = (0, 0)$ according to the following table:

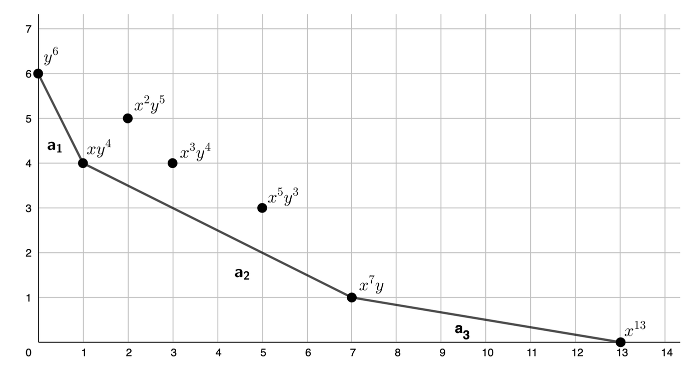
Point multiplicity	Number of tangents	Milnor Number	'Representative' curve
2	2	1	$A_1 : y^2 = x^2$
	1	$k \geq 2$	$A_k : y^2 = x^{k+1}$
3	3	4	$D_4 : x(y^2 - x^2) = 0$
	2	$k \geq 5$	$D_k : x(y^2 - x^{k-2}) = 0$
	1	6	$E_6 : x^3 + y^4 = 0$
		7	$E_7 : x^3 + xy^3 = 0$
		8	$E_8 : x^3 + y^5 = 0$
		> 8	?

Hence, we have that:

- the problem is solved for $m_P(\mathcal{C}) = 2$.
- if $m_P(\mathcal{C}) = 3$, the case to be studied is when there is a unique (triple) tangent. How can these points be analyzed? Using the Newton-Puiseux algorithm.

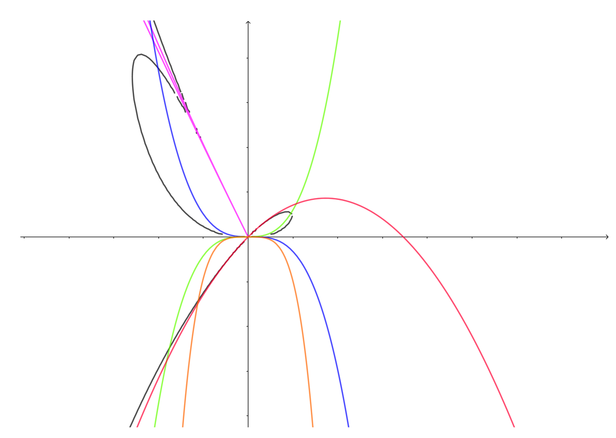
Newton-Puiseux algorithm

- A branch of a plane algebraic curve $\mathcal{C}$ at one of its points P can be thought as a local parameterization of $\mathcal{C}$ around P and is of the form $(p(T), q(T))$, $p(T), q(T) \in \mathbb{C}[[T]]$.
- A **Puiseux series** is an element of
$$\mathbb{C}\{\{x\}\} := \bigcup_{r=1}^{\infty} \mathbb{C}[[x^{\frac{1}{r}}]]$$
i.e. a power series in which positive rational exponents with a common denominator are admitted.
- The **Newton polygon** of a polynomial $f \in \mathbb{C}[x, y]$ is the convex hull of $\mathrm{supp}(f)$.



The Newton Polygon of $y^6 - 2xy^4 + x^7y - x^{13} + 3x^2y^5 - x^3y^4 + x^5y^3$

- Classically, the **Newton-Puiseux algorithm** is a procedure that allows to find a branch $(p(T), q(T))$ of $\mathcal{C}$ at P using Puiseux series and the Newton polygon. Nevertheless, we brought some modifications in order to find **all** the branches of $\mathcal{C}$ at P.
- In general, $p(T)$ and $q(T)$ have infinite terms but the **implicit function theorem (formal version)** allows us to understand when we can stop the algorithm.



Approximation of the branches of a curve using the N-P algorithm

Triple points with triple tangent

Adapting the procedure to the specific case of triple points with triple tangent, we obtain an algorithm that can be easily implemented in **CoCoA** or **Macaulay2**. If $\mathcal{C}$ is a curve with such a point P we have:

$$\left\{\begin{matrix}\textbf{Input:} \\ \textbf{Equation of } \mathcal{C},\ P\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\textbf{Output:} \\ \textbf{Approximations of the branches of } \mathcal{C} \textbf{ at } P\end{matrix}\right\}$$

Next goal: Using this method one can hope to find "representative" curves for triple points with a triple tangent, so completing the missing cases in the table above.

Hilbert Functions of aCM Veronese subvarieties

Joint work with: A. Boralevi[1], E. Carlini[1]

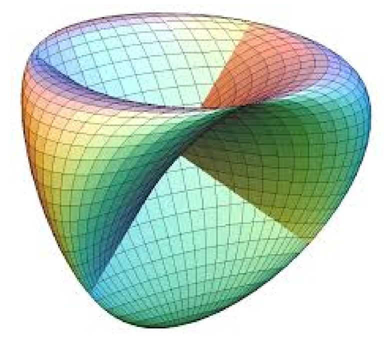
Question

If $\mathbb{X}$ is a subvariety of a Veronese variety, which is its Hilbert function?

General setting

Let $\nu_{n,d}$ be the Veronese map:

$$\nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{n}-1}, \qquad V_{n,d} = \nu_{n,d}(\mathbb{P}^n)$$



The (singular) embedding of $V_{2,2}$ in $\mathbb{P}^3$: the Steiner surface

We showed that, if $\mathbb{X} \subset V_{n,d}$ is a subvariety and we set $\mathbb{Y} = \nu_{n,d}^{-1}(\mathbb{X})$, then it holds that $(\mathcal{I}(\mathbb{X}))_t = (\mathcal{I}(\mathbb{Y}))_{dt}$. Hence we have

$$H_{\mathbb{X}}(t) = H_{\mathbb{Y}}(dt)$$

Main idea: Since $n << \binom{n+d}{n} - 1$ and $H_{\mathbb{X}}(t)$ is related to $H_{\mathbb{Y}}(t)$, a smart way to study the former is analyzing the latter.

The case of divisors

If $\mathbb{X}$ is a divisor, i.e. $\dim(\mathbb{X}) = n - 1$, it can be shown that

$$H_{\mathbb{X}}(t) : \begin{cases} \binom{n+t}{n}, & t \leq d-1 \\ \binom{n+t}{n} - \binom{n+t-d}{n}, & t \geq d \end{cases}$$

Clearly, since $\mathrm{codim}_{V_{n,d}}(\mathbb{X}) = 1$, this is the easiest case.

aCM Varieties

- A ring R is said to be **Cohen-Macaulay** if $\mathrm{depth}(\mathfrak{p}) = \mathrm{codim}(\mathfrak{p})$ for every maximal ideal $\mathfrak{p}$ of R.
- A projective variety $\mathbb{X}$ is said to be **geometrically Cohen-Macaulay** if $\mathcal{O}_{\mathbb{X},P}$ is a Cohen-Macaulay ring for each $P \in \mathbb{X}$.
- A projective variety $\mathbb{X}$ is said to be **arithmetically Cohen-Macaulay (aCM)** if its coordinate ring is Cohen-Macaulay.

If $\mathbb{X}$ is aCM, it has a good behaviour with respect to **hyperplane sections**: for example $\Delta^s H_{\mathbb{X}}(t)$ is still an Hilbert function.

Theorem

Using the fact that reduced 0-dimesional varieties are always aCM together with some properties of Hilbert functions of artinian ideals we proved the following theorem:

Hilbert functions of points on 2-dimensional Veronese variety

Let $(h_t)_{t\in\mathbb{N}}$ be the Hilbert function of a finite set of reduced points of $\mathbb{P}^{\binom{2+d}{2}-1}$ and let us set

$$t_1 = \max\left\{t \mid H_{\mathbb{X}}(t) = H_{V_{2,d}}(t)\right\} \qquad t_2 = \min\left\{t \mid H_{\mathbb{X}}(t) = |\mathbb{X}|\right\}$$

Then $\exists\, \mathbb{X} \subset V_{2,d} \subset \mathbb{P}^N$ such that $H_{\mathbb{X}}(t) = h_t$ if and only if the following conditions hold:

- $d(t_1 + 1) \geq \left\lceil \frac{\Delta H_{\mathbb{X}}(t_1+1)}{d} \right\rceil$
- $\left\lceil \frac{\Delta H_{\mathbb{X}}(t)}{d} \right\rceil \geq \left\lceil \frac{\Delta H_{\mathbb{X}}(t+1)}{d} \right\rceil \qquad \forall\, t_1 + 1 \leq t \leq t_2 - 1$

In other words, this theorem characterizes the Hilbert functions of points on $V_{2,d}$ for each $d \in \mathbb{N}$.

Next goal: Generalizing this theorem to codimension 2 aCM varieties on $V_{n,d}$.

Complete intersection on Veronese varieties

One may ask another question: are there complete intersection subvarieties on Veronese varieties? What, if any, are they?

So far we showed that:

- All and only c.i. subvarieties on $V_{1,d} \subset \mathbb{P}^d$ (i.e. on the rational normal curves) are the trivial ones, that is sets of one or two points.
- All and only c.i. subvarieties on $V_{2,2}$ are the conics and the sets of one or $2a$ points for each $a \in \mathbb{N}$.

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