Some Geometric Properties of Lasserre's Truncated Moments

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Poema

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Motivation: Polynomial Optimization

- $f, g_1, \ldots, g_r \in \mathbb{R}[\mathbf{X}] = \mathbb{R}[X_1, \ldots, X_n]$;
- $S = \mathcal{S}(\mathbf{g}) = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \quad \forall i\}$.

Goal of Polynomial Optimization: find

$$\begin{aligned} f^* &= \inf \{ f(x) \in \mathbb{R} \mid x \in S \} \\ &= \sup \{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathrm{Pos}(S) \} \\ &= \inf \{ \textstyle\int f \, d\mu \in \mathbb{R} \mid \mu \in \mathcal{M}^{(1)}(S) \}. \end{aligned}$$

General, NP hard problem: we need to develop strategies to find approximations of f^*: in particular approximations of $\mathrm{Pos}(S)$ and $\mathcal{M}(S)$.

Ingredients

- Σ^2 **Sum of Squares polynomials** (SoS);
- $Q = \mathcal{Q}(\mathbf{g}) = \Sigma^2 + \Sigma^2 \cdot g_1 + \cdots + \Sigma^2 \cdot g_r$ **quadratic module**;
- $\mathcal{L}(\mathbf{g}) = \{ \sigma \in \mathbb{R}[\mathbf{X}]^* \mid \forall q \in \mathcal{Q}(\mathbf{g}) \ \langle \sigma | q \rangle \geq 0 \}$;
- $\mathcal{M}(S)$ **Radon measures** supported on S;
- Conic duality: $(\cdot)^\vee$, notice that $\mathcal{M}(S) = \mathrm{Pos}(S)^\vee$ and $\mathcal{L}(\mathbf{g}) = Q^\vee$;
- Restriction to $\mathbb{R}[\mathbf{X}]_{\leq t}$: $(\cdot)^{[t]}$.

Lasserre Relaxations

At order d restrict to $\mathbb{R}[\mathbf{X}]_{\leq 2d}$ and replace:

- $\mathrm{Pos}(S)$ by Q;
- $\mathcal{M}(S)$ by $\mathcal{L}(\mathbf{g})$.

$$f^*_{\mathrm{MoM},d} = \inf \{ \langle \sigma | f \rangle \in \mathbb{R} \mid \sigma \in \mathcal{L}_{2d}(\mathbf{g}), \ \langle \sigma | 1 \rangle = 1 \}$$

Our Goal

Describe and study the convex cones

$$\mathcal{L}_{2d}(\mathbf{g}) = \{ \sigma \in \mathbb{R}[\mathbf{X}]^* \mid \forall q \in \mathcal{Q}_{2d}(\mathbf{g}) \ \langle \sigma | q \rangle \geq 0 \}$$

(and its affine generating section $\mathcal{L}^{(1)}_{2d}(\mathbf{g})$) when restricted to $\mathbb{R}[\mathbf{X}]_{\leq t}$. In particular:

- **geometric properties** of $\mathcal{L}_{2d}(\mathbf{g})^{[t]}$;
- **convergence** of $\mathcal{L}_{2d}(\mathbf{g})^{[t]}$ as $d \to \infty$;
- relationship with the **geometry of** S.

Special Guests: Evaluations & Veronese Embedding

- $\mathbf{e}_x$ denotes the **evaluation** at $x \in \mathbb{R}^n$:
$$\langle \mathbf{e}_x | f \rangle = \int f \, d\mathbf{e}_x = f(x)$$
- Richter–Tchakaloff theorem:
$$\mathcal{M}(S)^{[t]} = \mathrm{cone}(\mathbf{e}_x \colon x \in S)^{[t]}$$

We can describe **all** the measures supported on S acting on $\mathbb{R}[\mathbf{X}]_{\leq t}$ using only evaluations!

Remark: If we express $\mathbf{e}_x^{[t]} \in (\mathbb{R}[\mathbf{X}]_t)^*$ using the dual of the monomial basis, we obtain: $\mathbf{e}_x^{[t]} = (1, x_1, \ldots, x_1^2, x_1x_2, \ldots, x_n^t)$, i.e.:

$$\begin{aligned} \mathrm{Ver}_t \colon S &\to \mathcal{M}^{(1)}(S)^{[t]} \subset (\mathbb{R}[\mathbf{X}]_{\leq t})^* \\ x &\mapsto \mathbf{e}_x^{[t]} \end{aligned}$$

and $\mathcal{M}(S)$ is the convex conve over $\mathrm{Ver}(S)$.

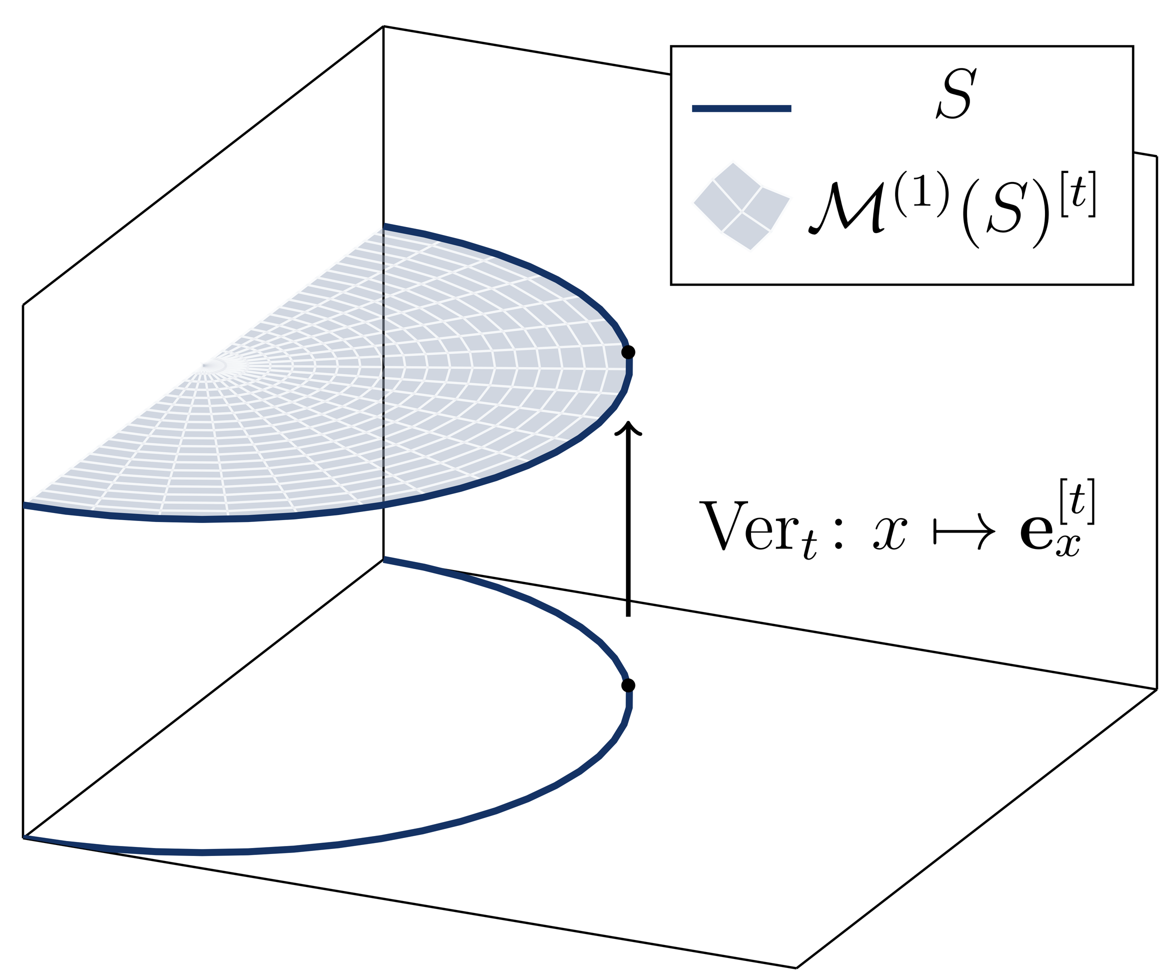

Figure: Veronese embedding

The Outer Aproximation $\mathcal{L}_{2d}(\mathbf{g})^{[t]}$

The truncated positive linear functionals $\mathcal{L}_{2d}(\mathbf{g})^{[t]}$:

- are closed convex cones (**spectrahedra**);
- are outer approximations of the measures: $\mathcal{M}(S) \subset \mathcal{L}_{2d}(\mathbf{g})^{[t]}$;
- have $\mathbf{e}_x^{[t]} \colon x \in S$ as (part of its) **extremal rays**: a supporting hyperplane containg $\mathbf{e}_x^{[t]}$ is defined by the polynomial $\sum_i (X_i - x_i)^2$.

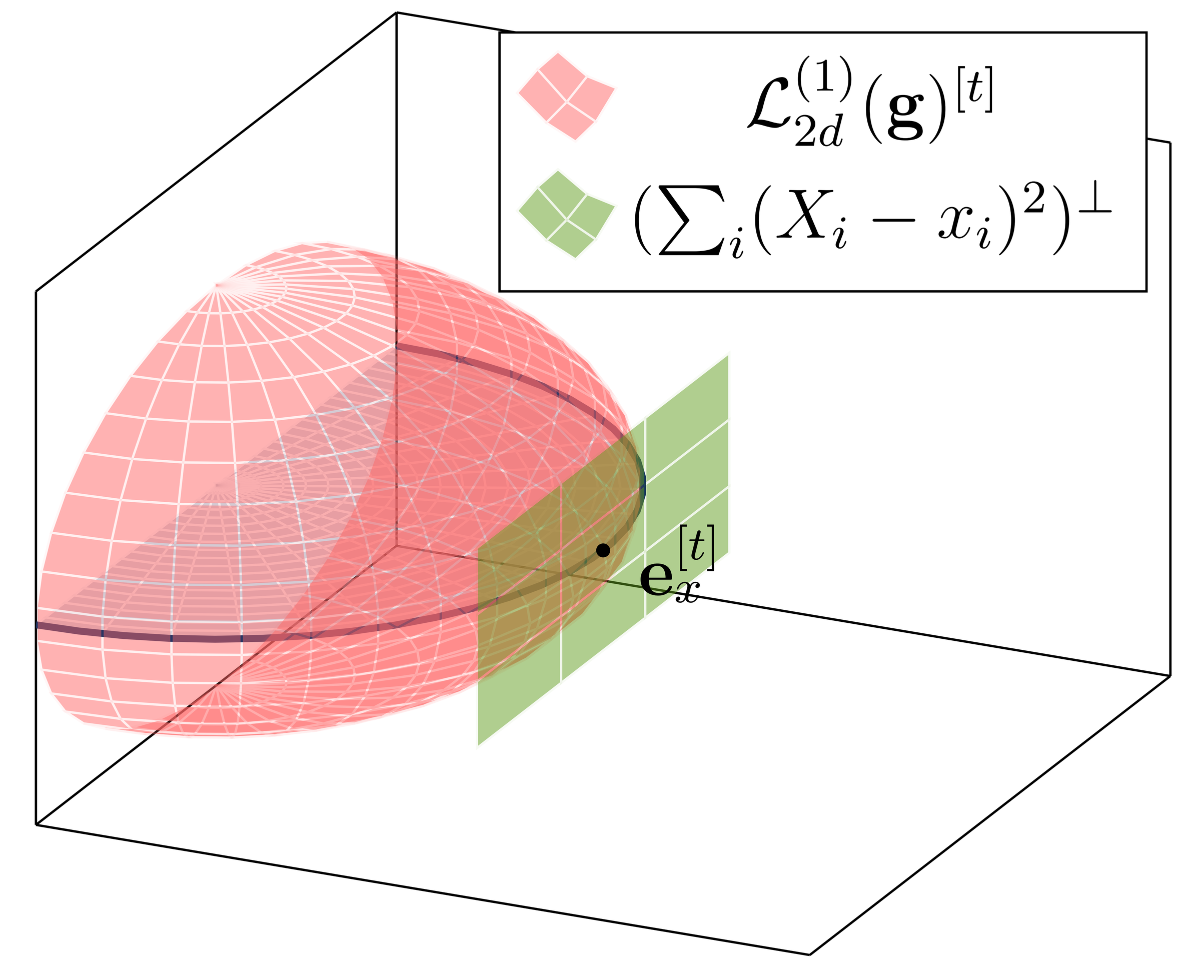

Figure: Outer approximation and supporting hyperplane

Convergence Properties

Archimedean property: $r^2 - (X_1^2 + \cdots + X_n^2) \in Q$ (compactness of S)

- **If Q is Archimedean the outer approx. converge to the measures**:
$$\bigcap_{d=t}^{\infty} \mathcal{L}_d(\mathbf{g})^{[t]} = \mathcal{M}(S)^{[t]}.$$
- order of convergence from effective versions of Putinar's Positivstellensatz;
- **if S is finite** (and $\dim \frac{\mathbb{R}[\mathbf{X}]}{Q \cap -Q} = 0$) **the convergence is finite** and the degree depends on the Castelnuovo-Mumford regularity of S.

Corollary: For a generic MoM relaxation, in degree big enough the minimizing linear functionals are sums of evaluations at the minimizers.

Duality

- $\mathbb{R}[\mathbf{X}]^*$ is an $\mathbb{R}[\mathbf{X}]$-module: for $\sigma \in \mathbb{R}[\mathbf{X}]^*$ and $h \in \mathbb{R}[\mathbf{X}]$
$$h \star \sigma = \sigma \circ m_h, \text{ i.e. } \langle h \star \sigma | f \rangle = \langle \sigma | f h \rangle .$$
- $\mathrm{Ann}_d(\sigma)$: annihilator in $\mathbb{R}[\mathbf{X}]_{\leq d}$.

Generic Linear Functionals

Let $x, y \in \mathbb{R}^n$. Then:

$$\begin{aligned} \mathrm{Ann}(\mathbf{e}_x + \mathbf{e}_y) &= \{ h \in \mathbb{R}[\mathbf{X}] \mid h \star (\mathbf{e}_x + \mathbf{e}_y) = 0 \} \\ &= \mathrm{Ann}(\mathbf{e}_x) \cap \mathrm{Ann}(\mathbf{e}_y) = \mathcal{I}_\mathbb{R}(x) \cap \mathcal{I}_\mathbb{R}(y). \end{aligned}$$

So we need the smallest annihilator to describe all the points $x \in S$: $\sigma^* \in \mathcal{L}_{2d}(\mathbf{g})$ is **generic** if

$$\mathrm{Ann}_d(\sigma^*) = \bigcap_{\sigma \in \mathcal{L}_{2d}(\mathbf{g})} \mathrm{Ann}_d(\sigma).$$

- Relative interior point $\Rightarrow$ generic point.

Genericity and Geometry of S

Using the Real Nullstellensatz, for d big enough (and discarting the highest degree terms of σ^*):

- $(\mathrm{Ann}_d(\sigma^*)) = \sqrt{Q \cap -Q}$;
- $(\mathrm{Ann}_d(\sigma^*)) = \mathcal{I}_\mathbb{R}(S)$ using the products of $\mathbf{g}$;
- $\mathrm{Ann}_d(\sigma^*) = \mathcal{I}_\mathbb{R}(\mathcal{V}_\mathbb{R}(\mathbf{h}))$ (equalities).

Remarks and Perspectives

- Exploit structure and symmetry of S and $\mathbf{g}$ to describe $\mathcal{L}_{2d}(\mathbf{g})$.
- Stopping criterion for $(\mathrm{Ann}_d(\sigma^*))$ being a real radical ideal.
- Exploit low rank structure of $\mathrm{Ann}_d(\mu)$ for $\mu \in \mathcal{M}(S)$ to improve Lasserre relaxations.

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