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The PageRank Computation in Google, Randomized Algorithms and Consensus of Multi-Agent Systems

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This talk

- The objective of this talk is to discuss three apparently unrelated topics

- Search engines (PageRank computation in Google)
- Randomized algorithms (Las Vegas type)
- Consensus of multi-agent systems

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- Search engines (PageRank computation in Google)
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- The math behind: theory of positive matrices

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- Search engines (PageRank computation in Google)
- Randomized algorithms (Las Vegas type)
- Consensus of multi-agent systems

- Multiple web page updates, web page aggregation, robustness for fragile links, web semantics

The PageRank Problem in Google

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Kobe University

PageRank for Kobe University (in Japanese)

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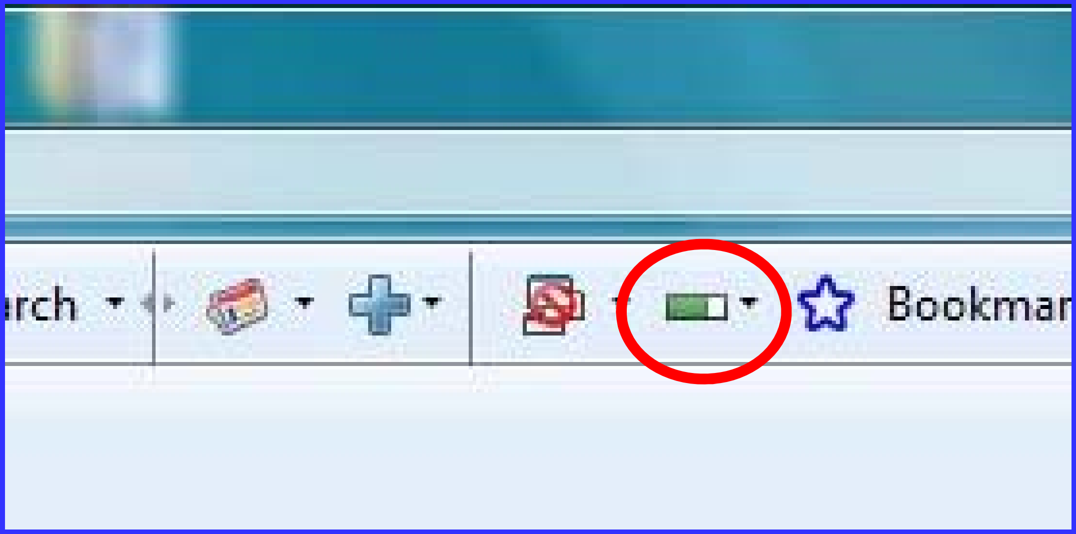

"PageRank is Google's view of the importance of this page (8/10)"

PageRank is a numerical value in the interval [0,1] which indicates the importance of the page you are visiting

PageRank for Kobe University (in Japanese)

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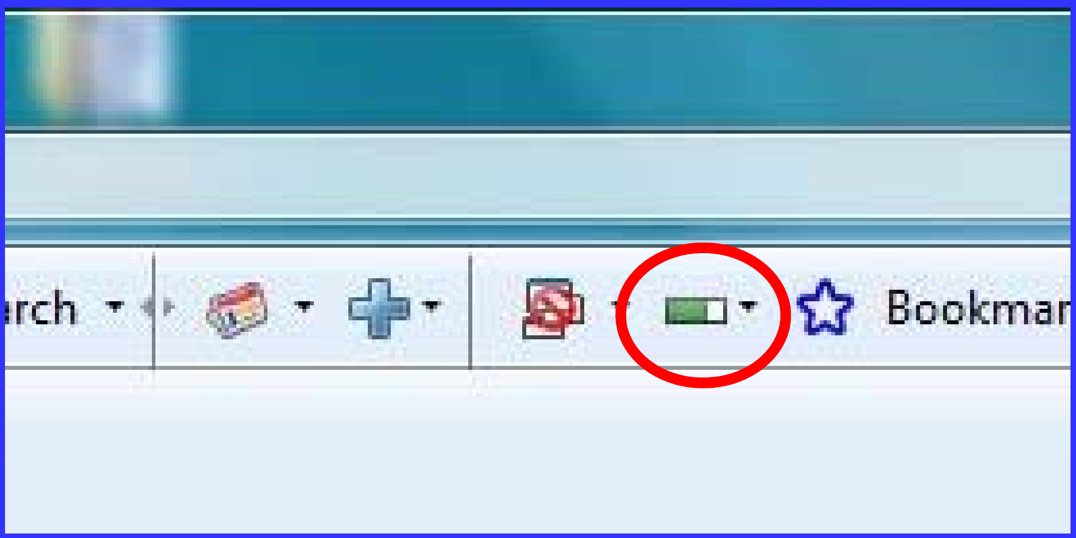

"PageRank is Google's view of the importance of this page (8/10)"

If you look at PageRank for Kobe University in English you get 7/10: The page is less visited

Random Surfer Model

- Web surfer moves along randomly following the hyperlink structure
- When arriving at a page with several outgoing links, one is chosen at random, then the random surfer moves to a new page, and so on…

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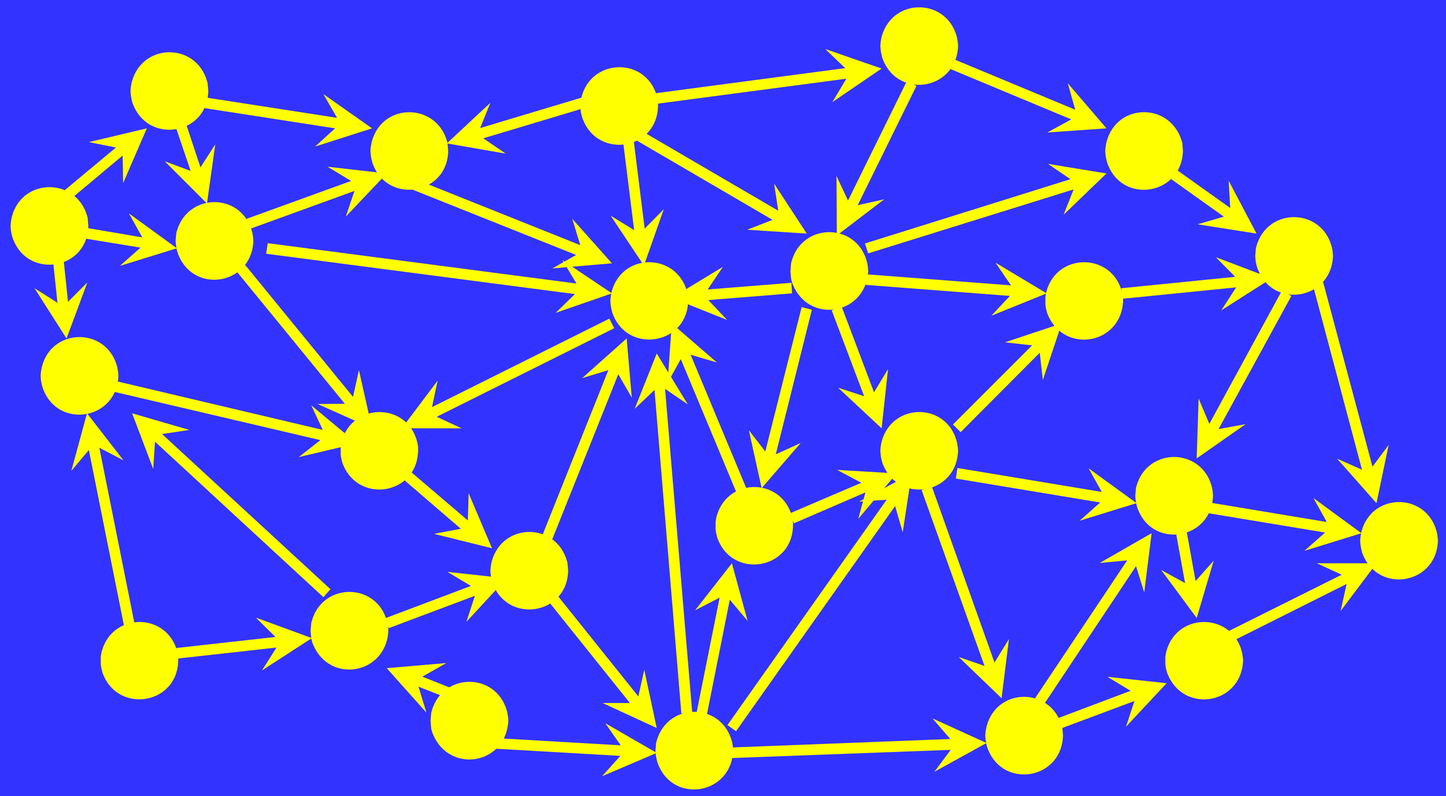
Random Surfer Model

- Web representation with incoming and outgoing links



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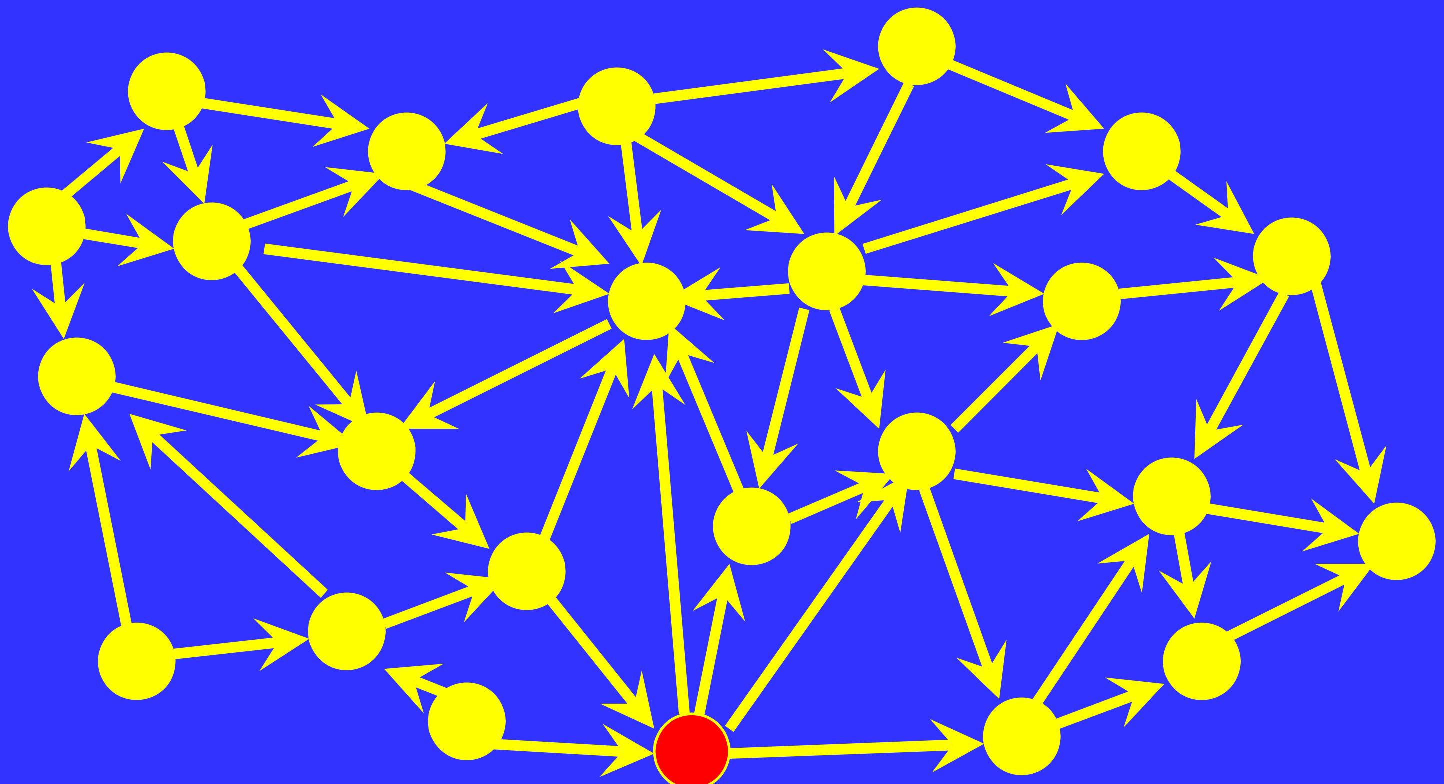
Random Surfer Model



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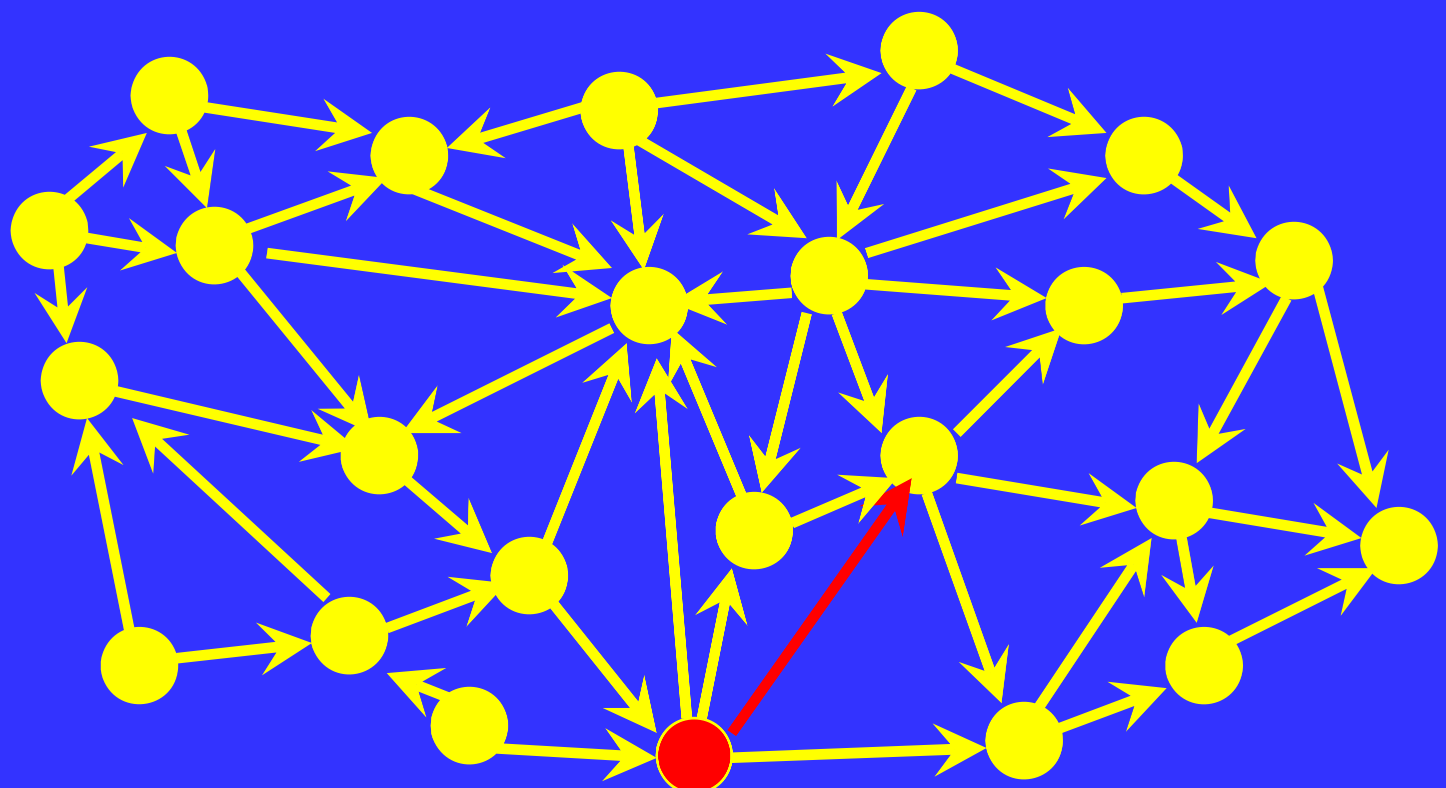
Random Surfer Model

- Pick an outgoing link at random



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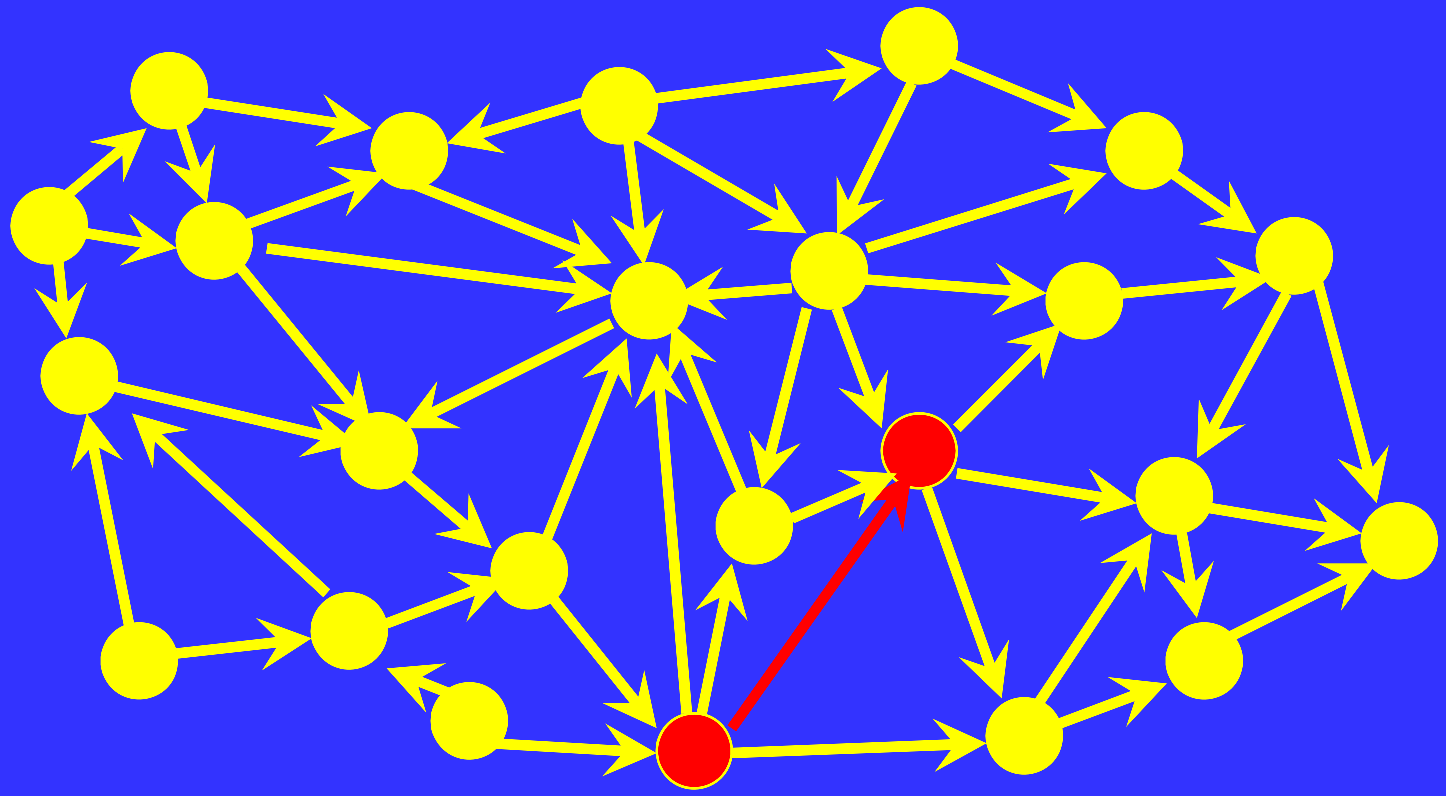
Random Surfer Model

- Arriving at a new web page



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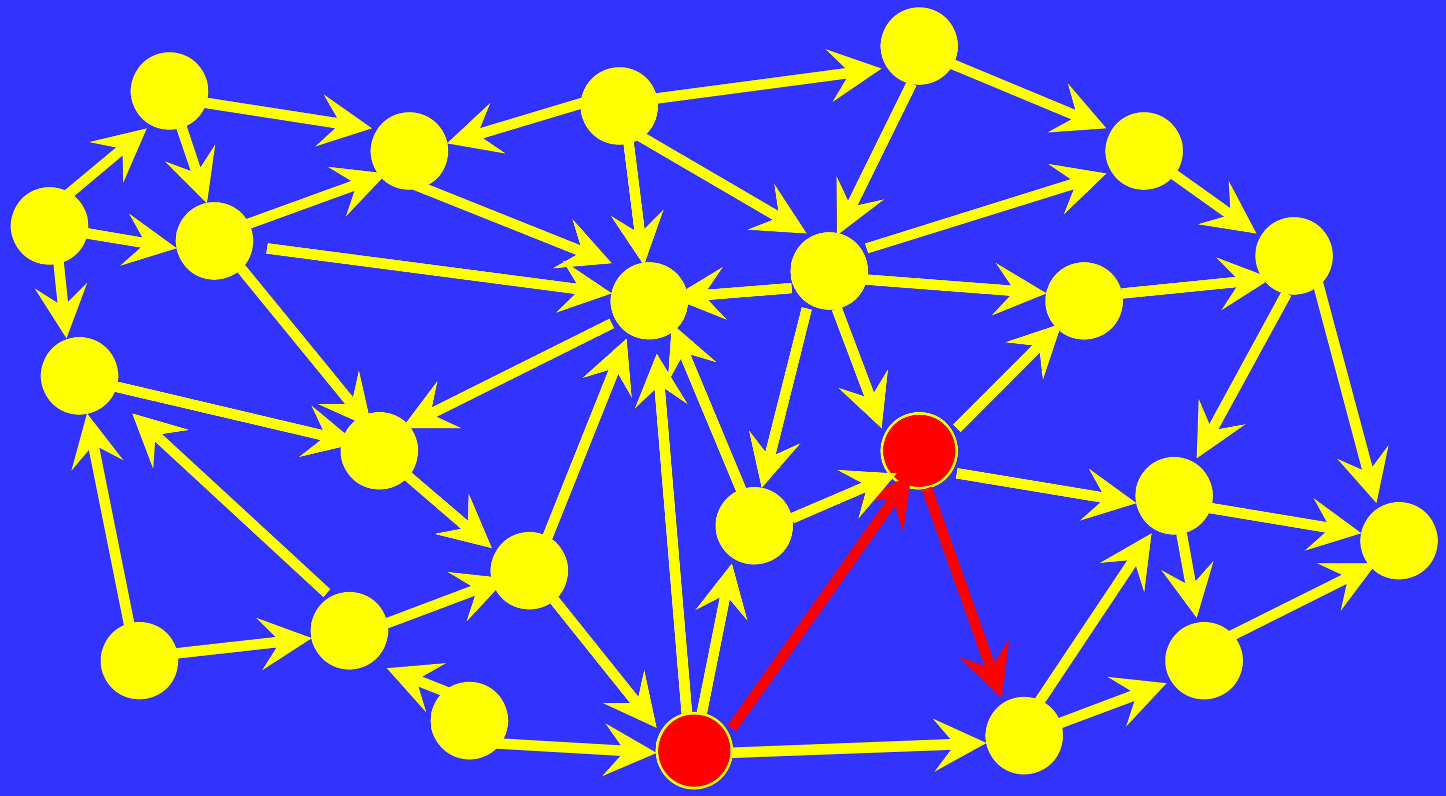
Random Surfer Model

- Pick another outgoing link at random



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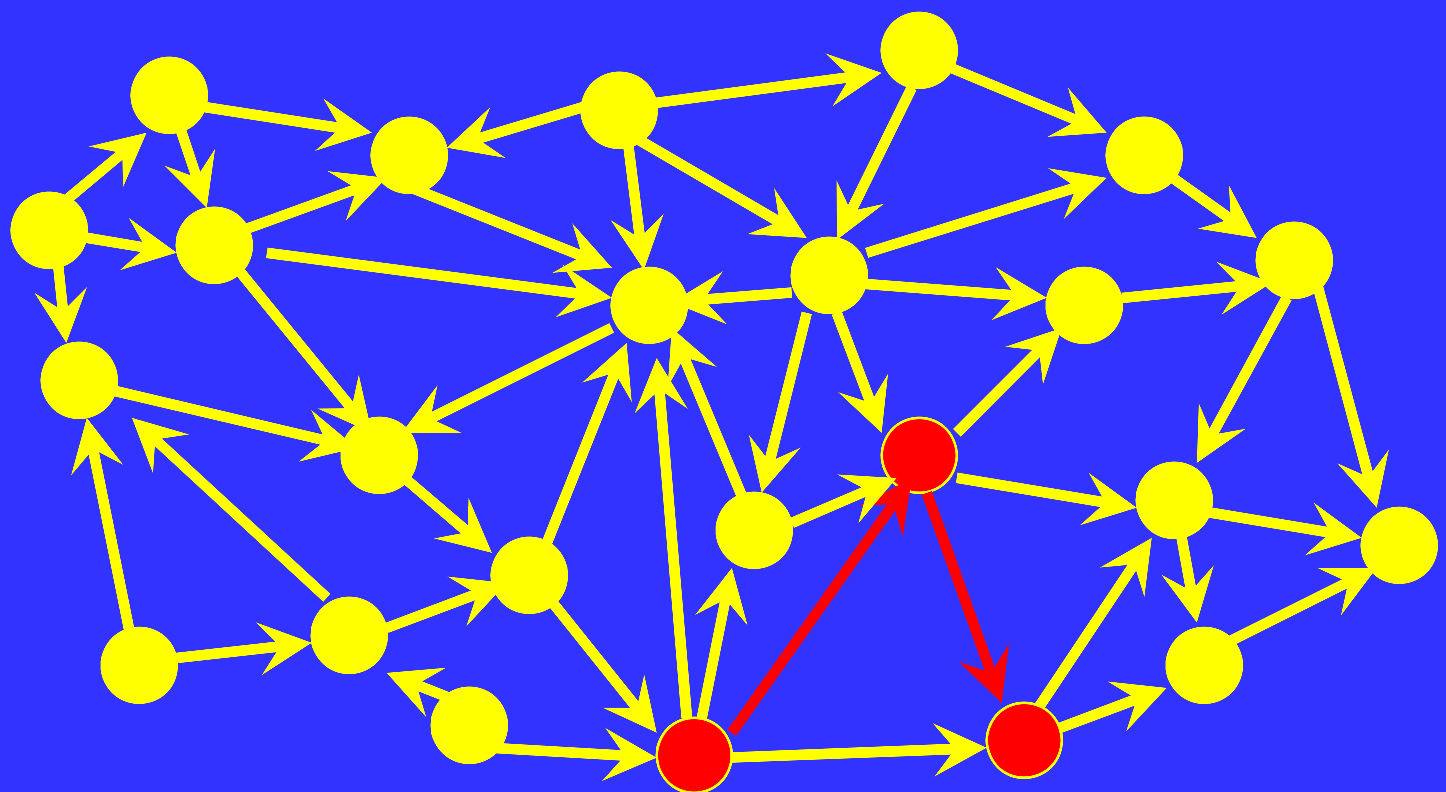
Random Surfer Model



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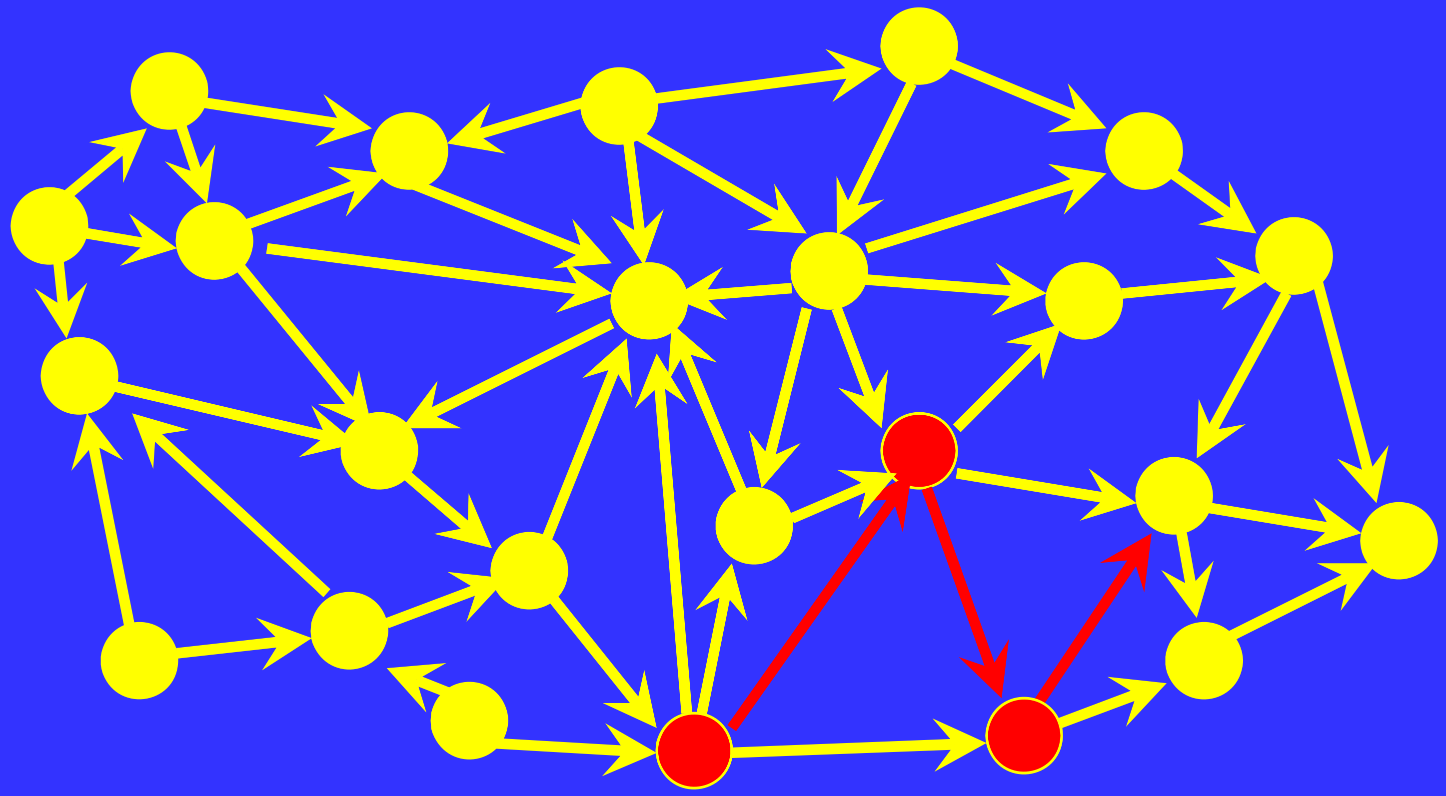
Random Surfer Model



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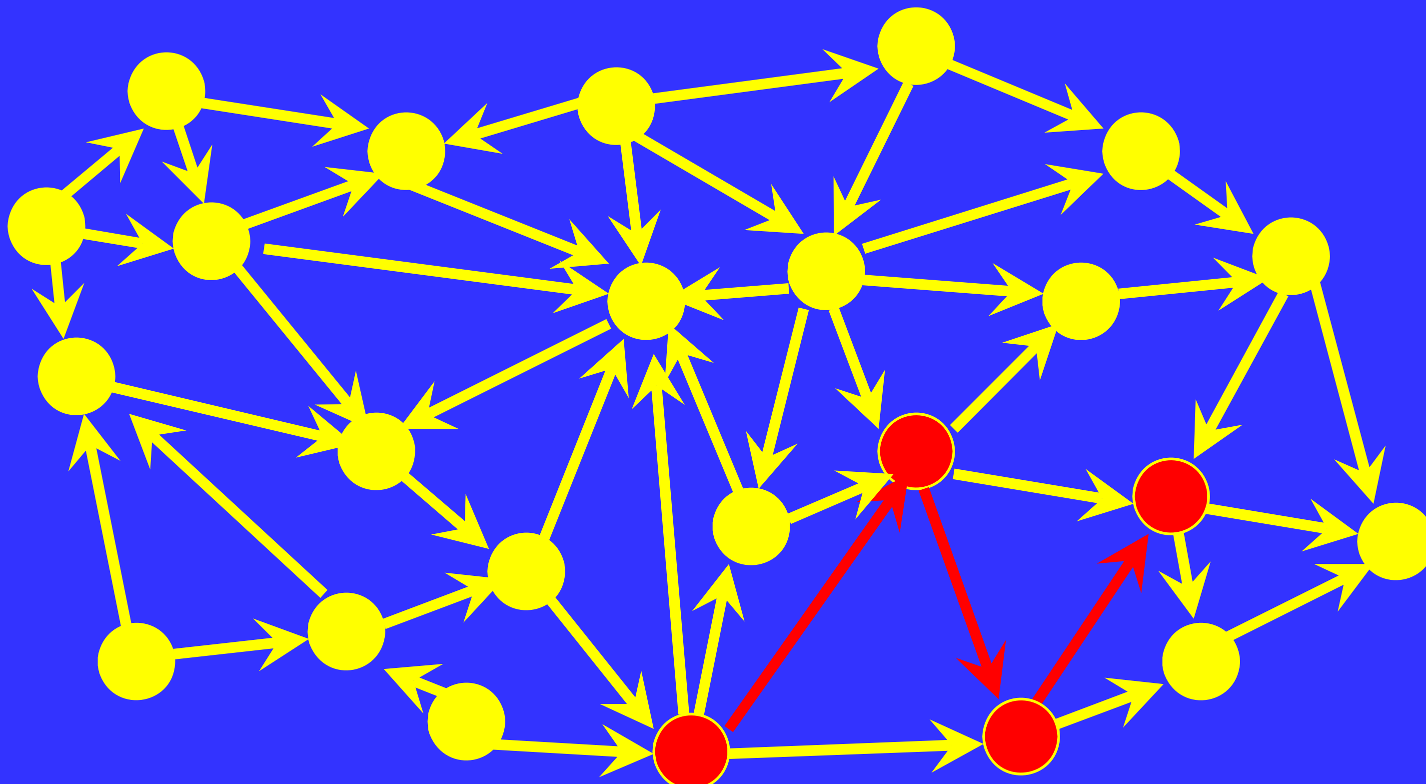
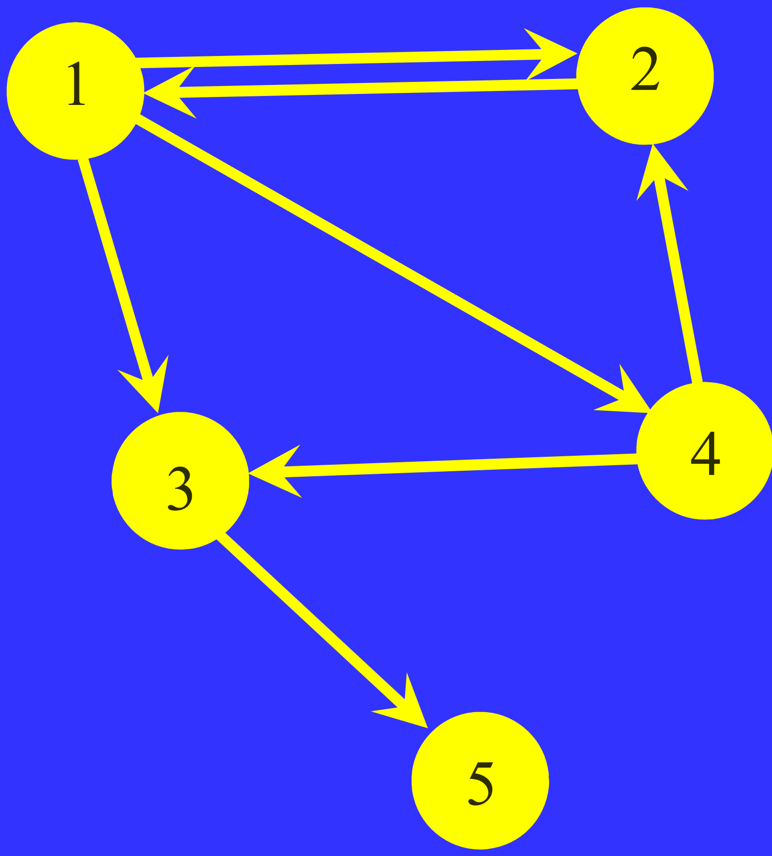
Random Surfer Model



Random Surfer Model

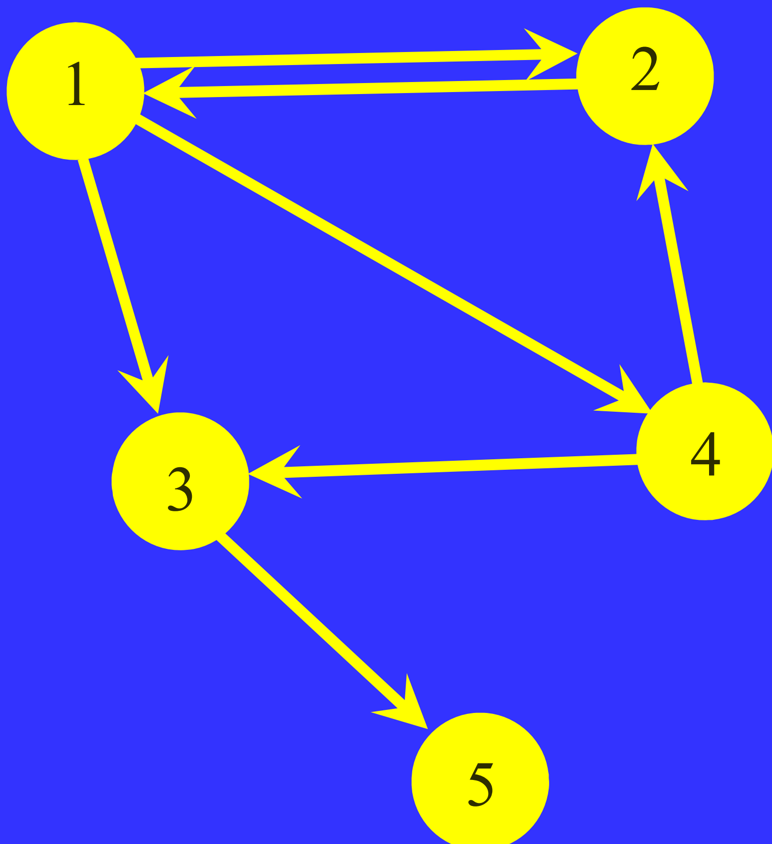
- If a page is “important” then it is visited more often...
- The time the random surfer spends on a page is a measure of the importance of the page
- If important pages point to your page, then your page becomes important (because it is often visited)
- For facilitating the web search we need to rank the pages
- We assign a numerical value to each page

Graph Representation



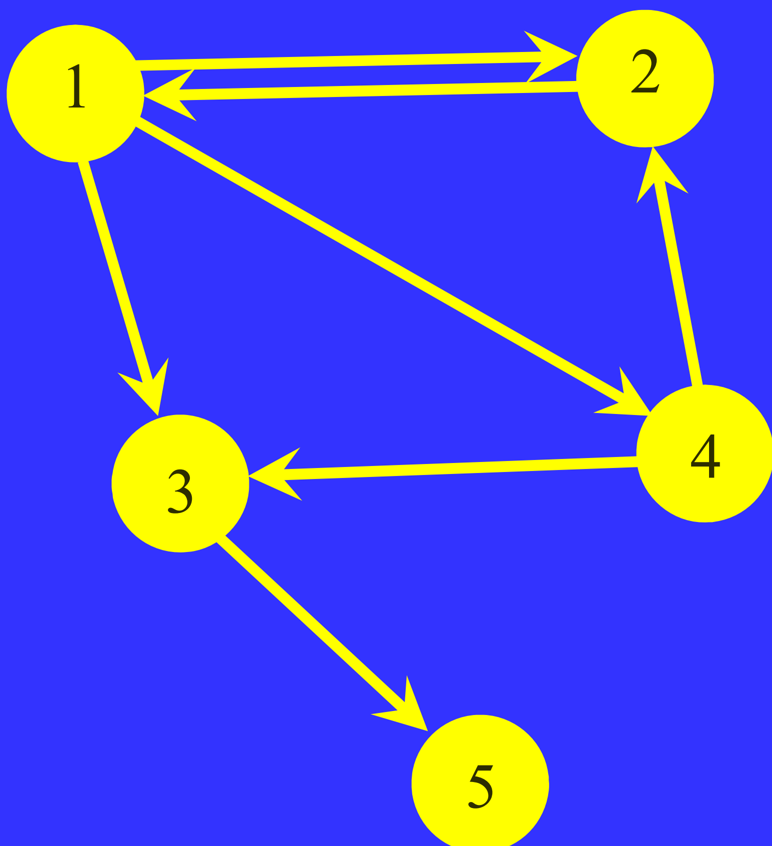
- Directed graph with nodes (pages) and links representing the web
- Graph is not necessarily strongly connected
- Graph is constructed using crawlers and spiders which move continuously along the web

Hyperlink Matrix



- For each node we count the number of outgoing links and normalize them to 1
- Hyperlink matrix is a nonnegative (column) substochastic matrix

Hyperlink Matrix



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

PageRank: Bringing Order to the Web[1,2]

- Need to rank pages in order of importance
- The PageRank x^* is defined as

$$x^* = Ax^* \quad \text{where} \quad x^* \in [0,1]^n \ \text{and} \ \sum_i x_i^* = 1$$

- x^* is a nonnegative unit eigenvector corresponding to the eigenvalue 1 for the hyperlink matrix A
- The question is when x^* exists and it is unique

[1] S. Brin, L. Page (1998)
[2] S. Brin, L. Page, R. Motwani, T. Winograd (1999)

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Issue of Dangling Nodes

- First issue: We have *dangling nodes*
- Random surfer gets “stuck” when visiting a pdf file
- In this case the “back button” of the browser is used
- Mathematically, the hyperlink matrix is nonnegative and (column) substochastic
- Easy fix: Add artificial links to make the matrix stochastic

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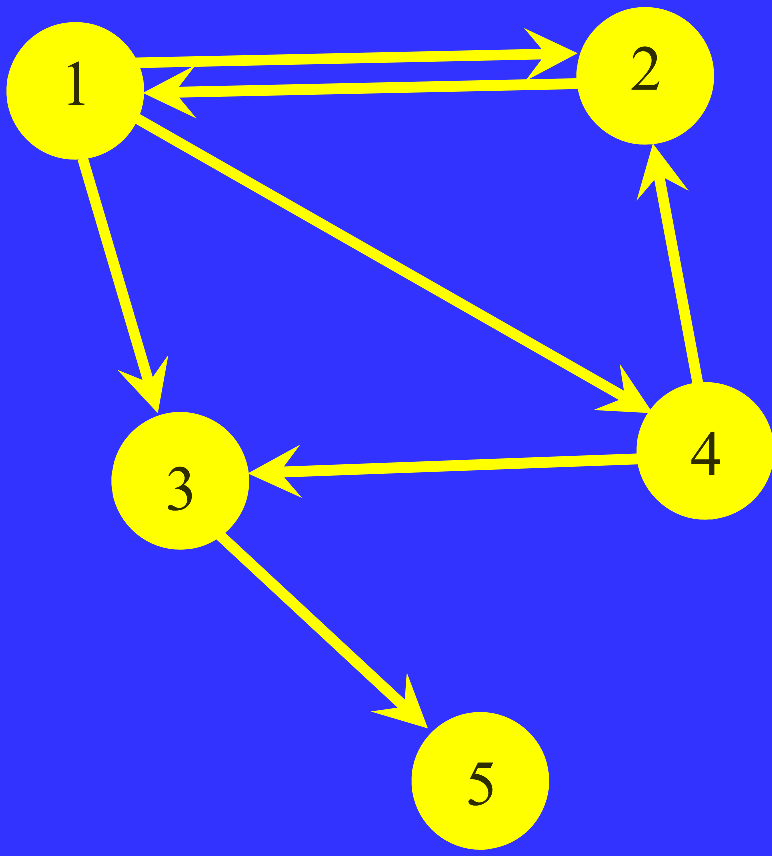
Page 5 is a Dangling Node



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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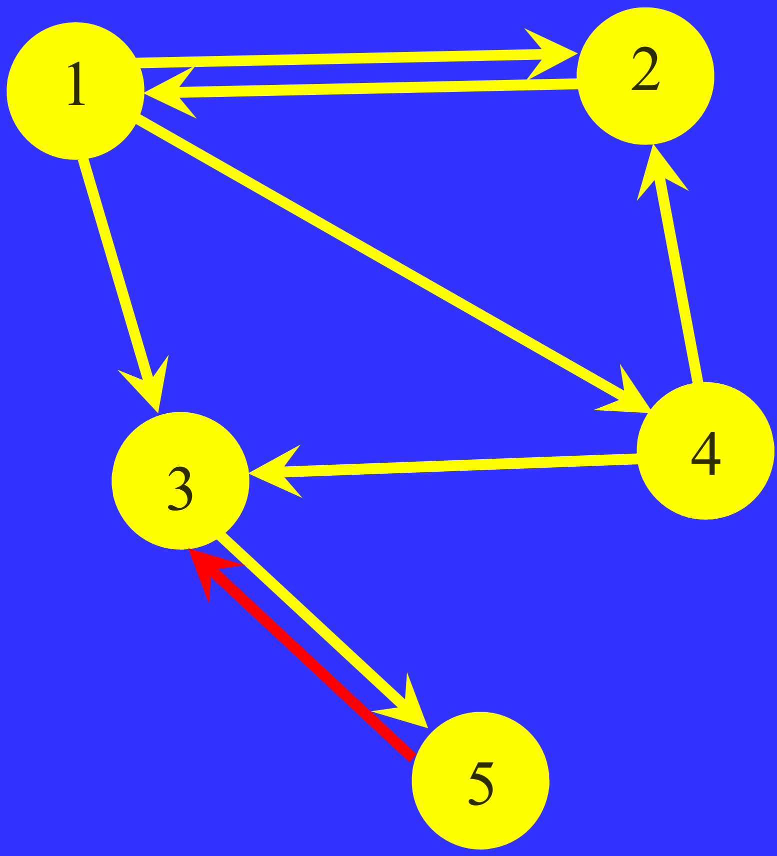
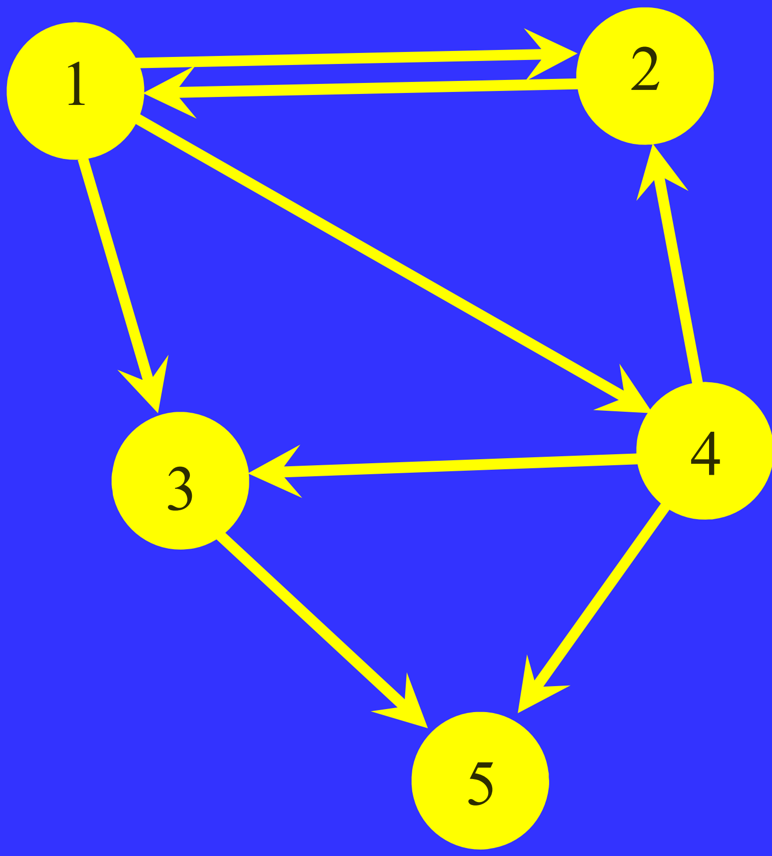
Easy Fix

- We add an outgoing link to page 5



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/2 & 1 \\ 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

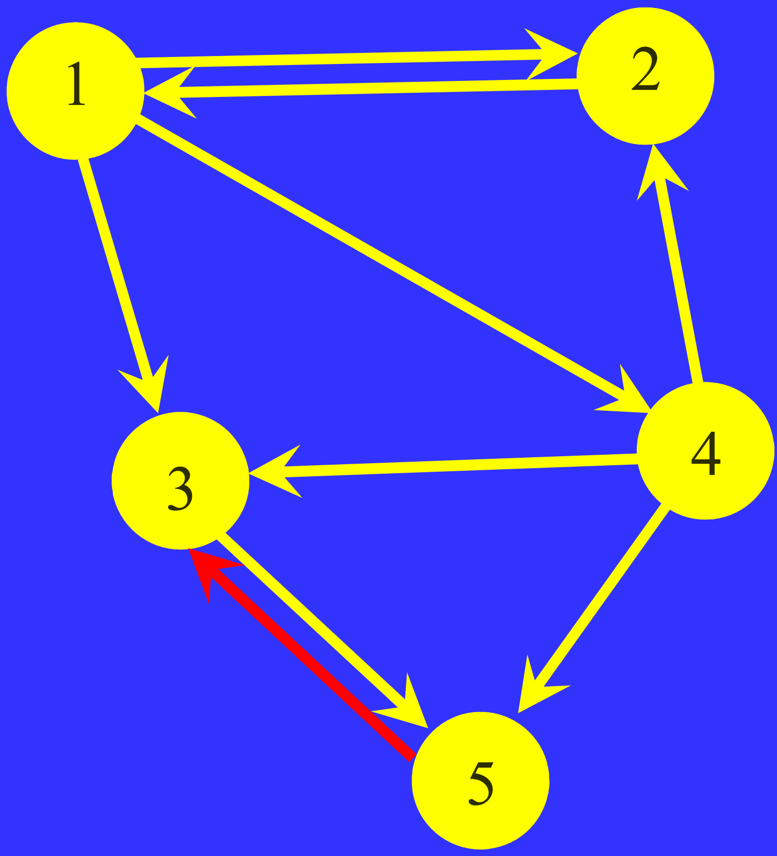
But in General the Fix is not so Easy…



- Page 5 has two incoming links

But in General the Fix is not so Easy…

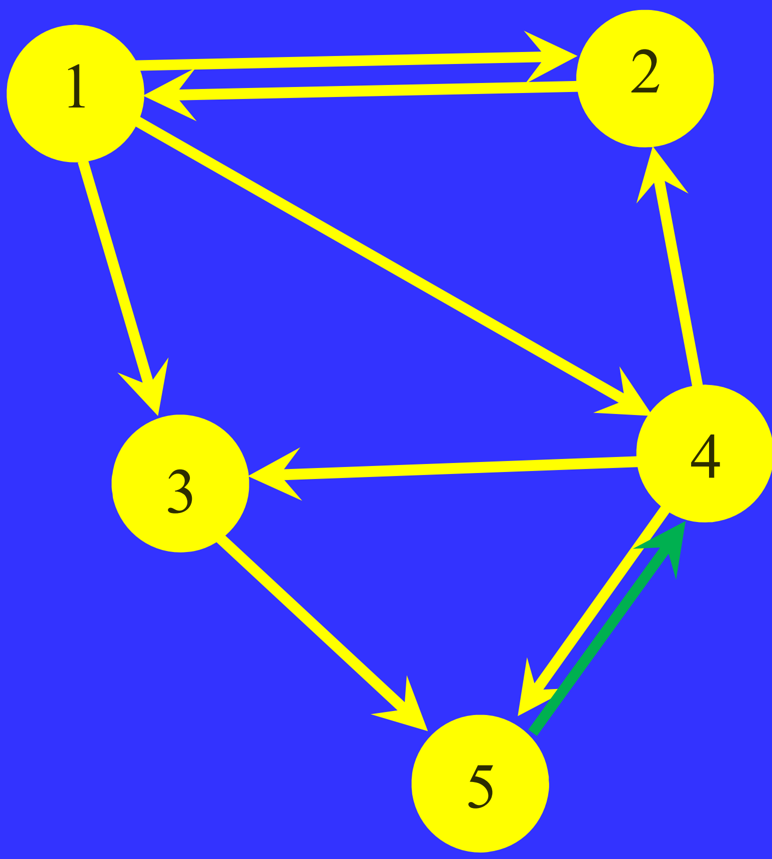
- We add an outgoing link from 5 to 3…



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1 \\ 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \end{bmatrix}$$

But in General the Fix is not so Easy…

- … or we add an outgoing link from 5 to 4?



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 \\ 1/3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/3 & 0 \end{bmatrix}$$

Modified Hyperlink Matrix

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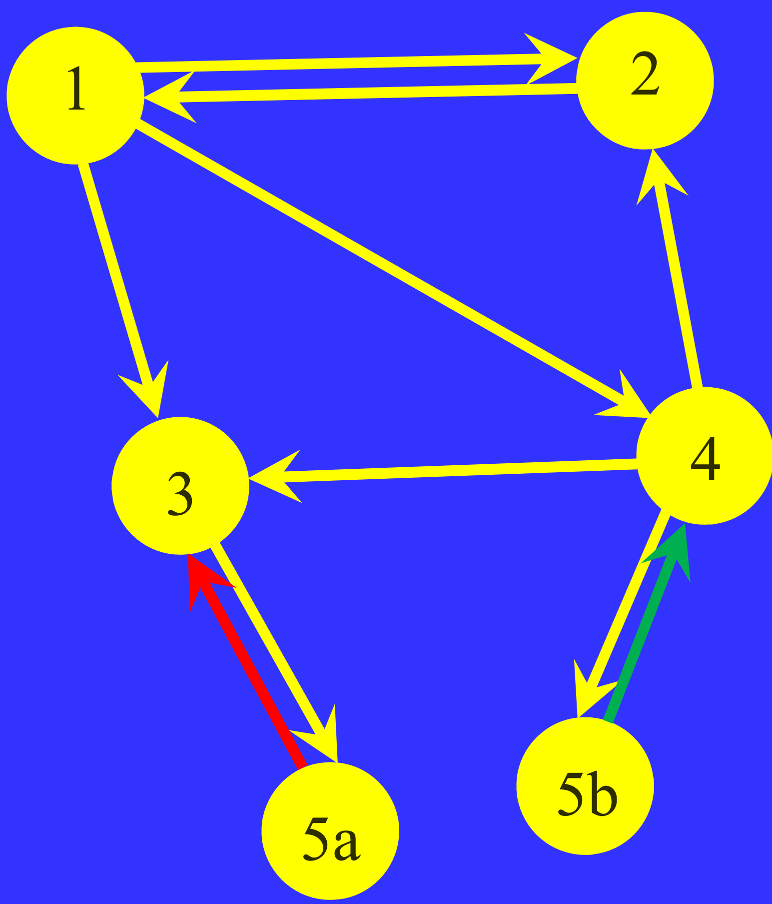

- A solution may be to break page 5 into two pages 5a and 5b
- This artificially changes the number of pages (not only the number of links) and the topology of the network

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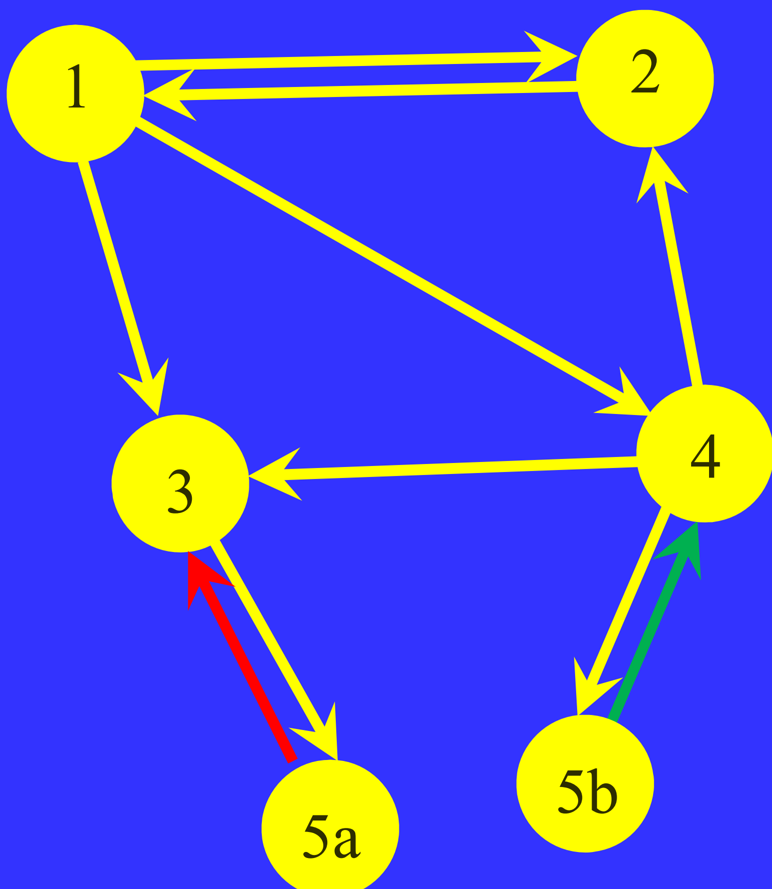
Modified Hyperlink Matrix



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \end{bmatrix}$$

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Assumption: No Dangling Nodes

- We assume that there are no dangling nodes
- This implies that A is a nonnegative stochastic matrix (instead of substochastic) having at least one eigenvalue equal to one
- **Second issue:** This eigenvalue is not necessarily unique

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Teleportation Matrix

- The random surfer may get bored after a while, and decides to “jump” to another page not directly connected to that currently visited

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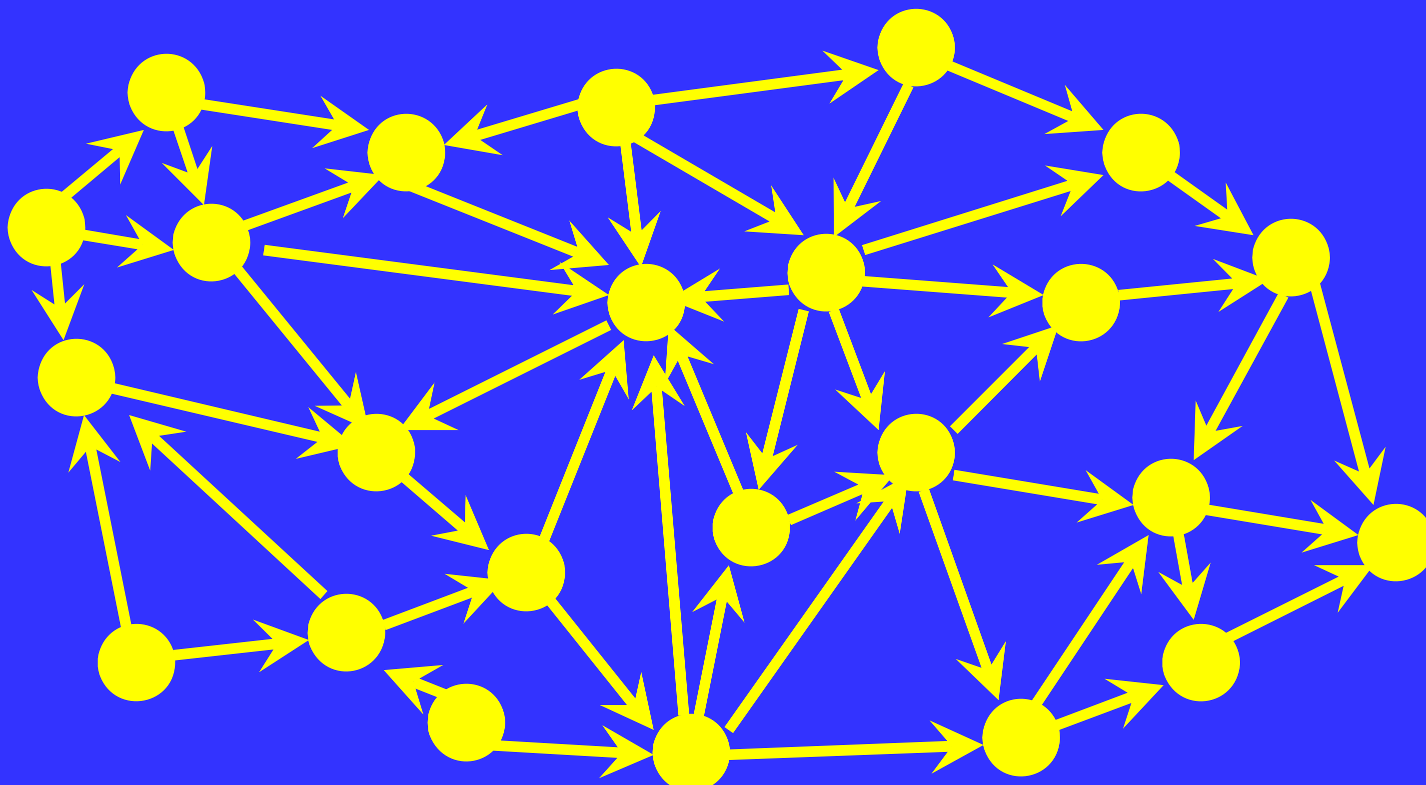
Recall the Random Surfer Model

- Web representation with incoming and outgoing links



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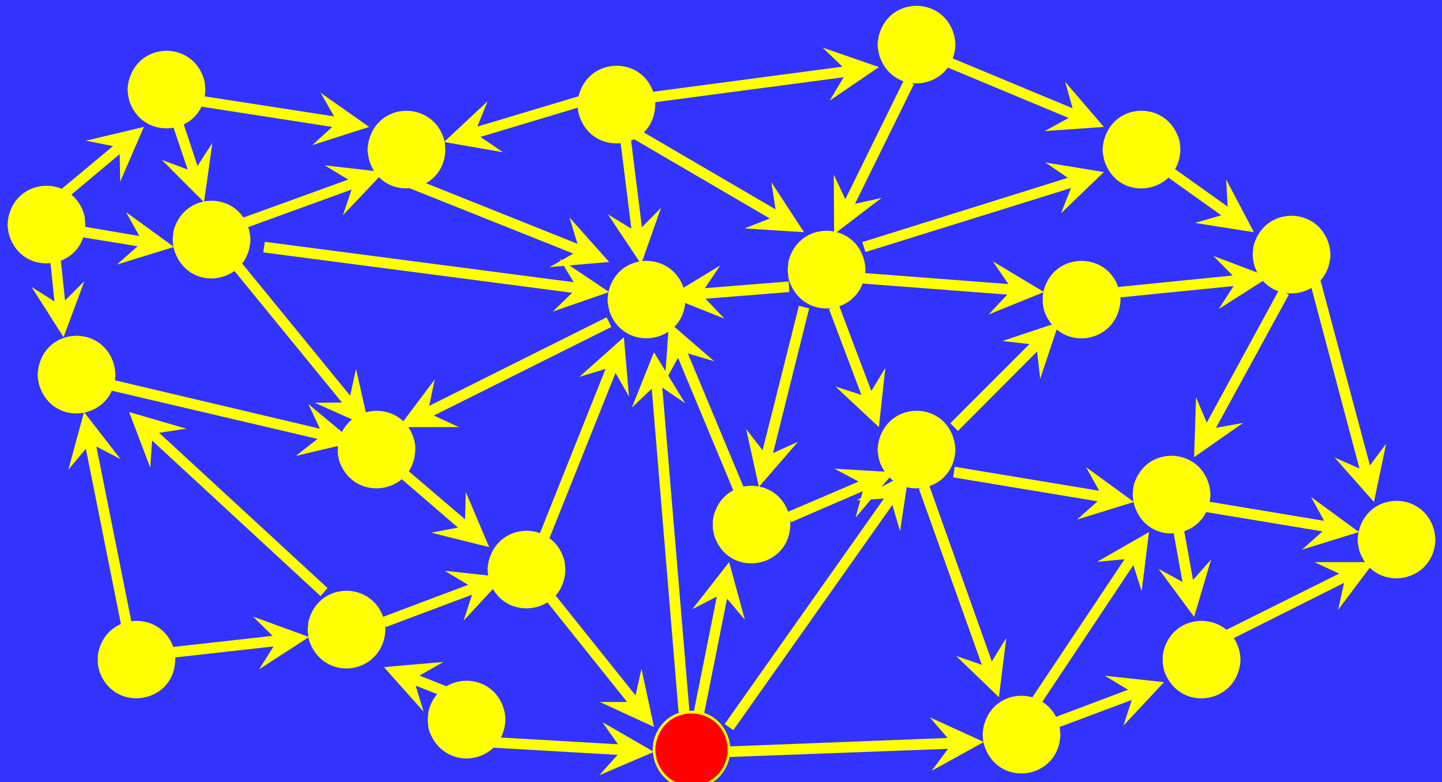

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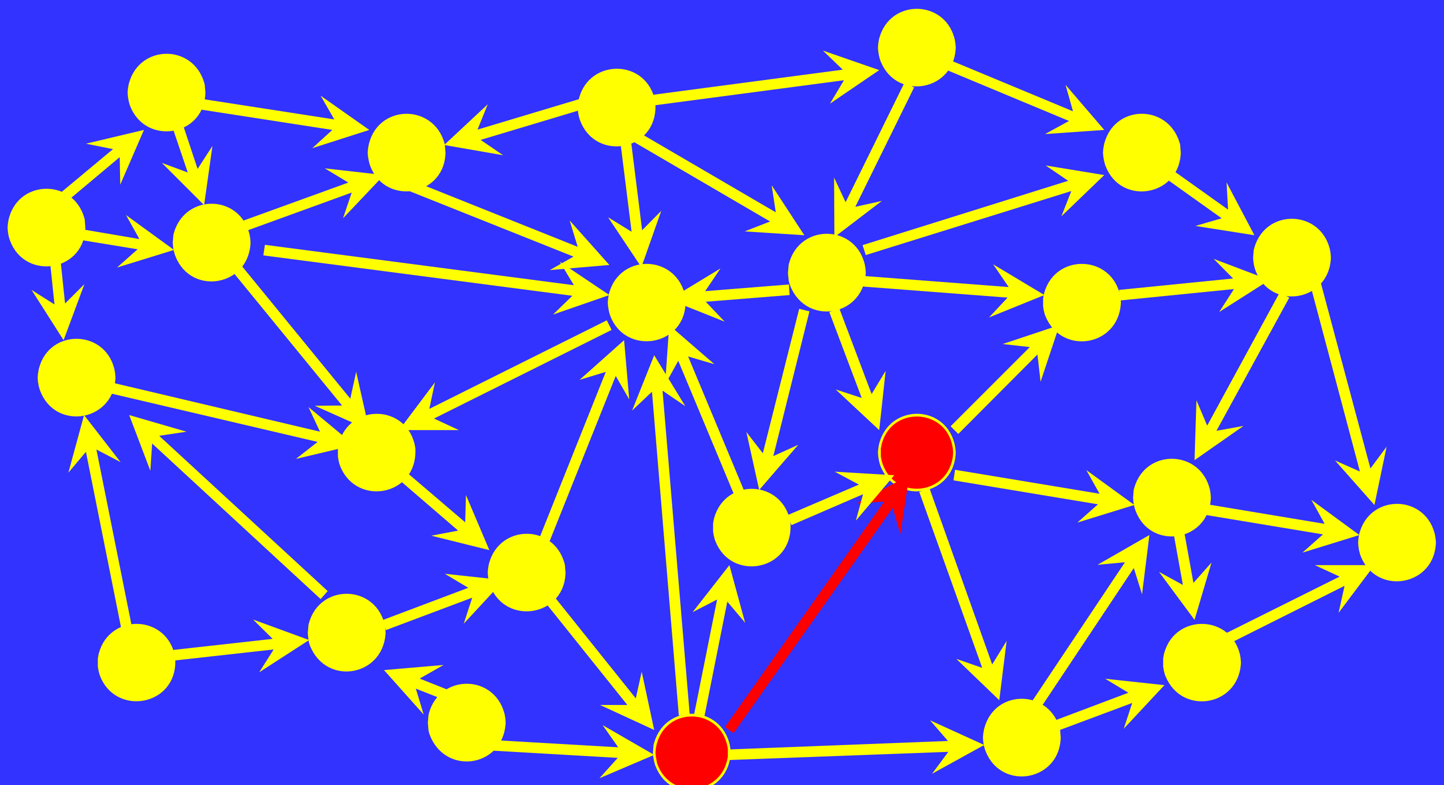

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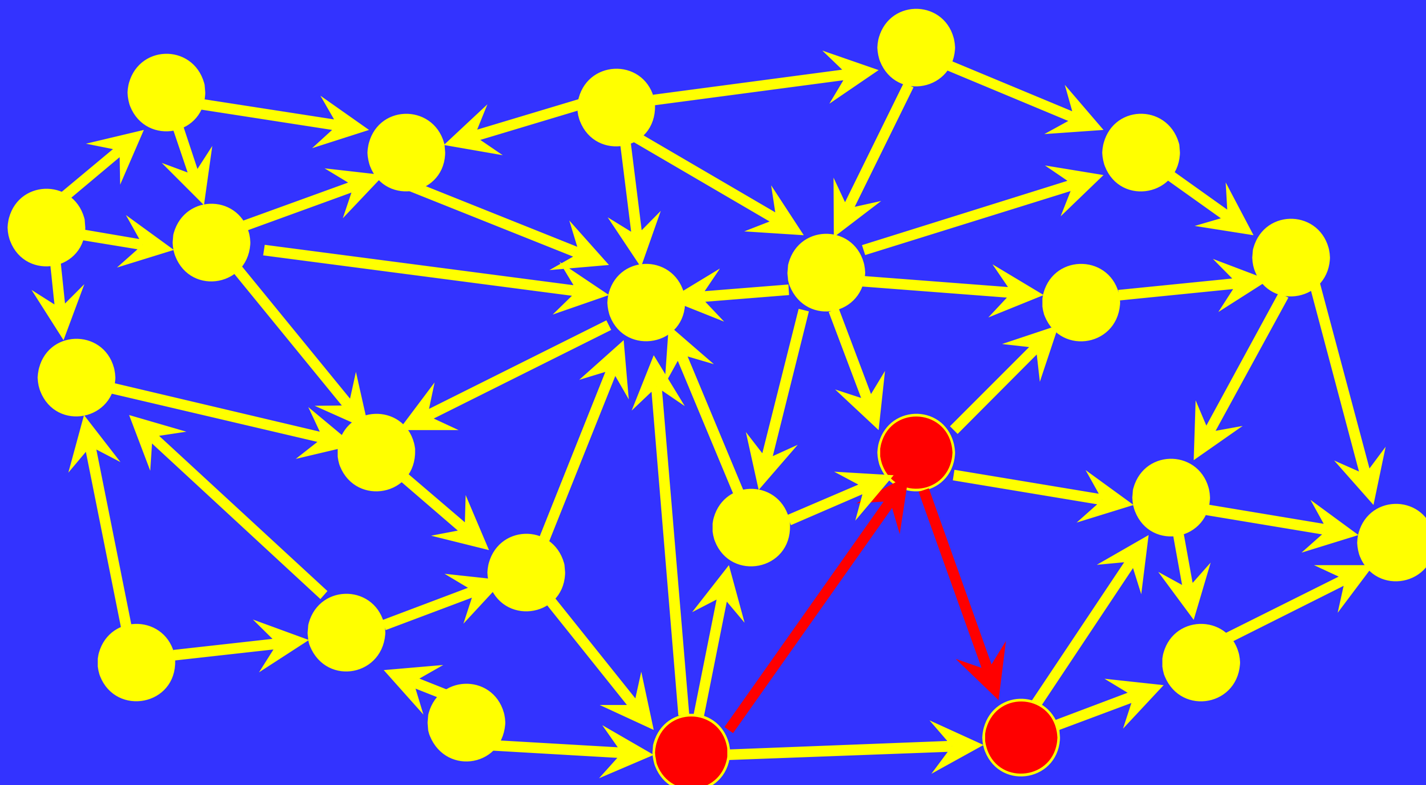

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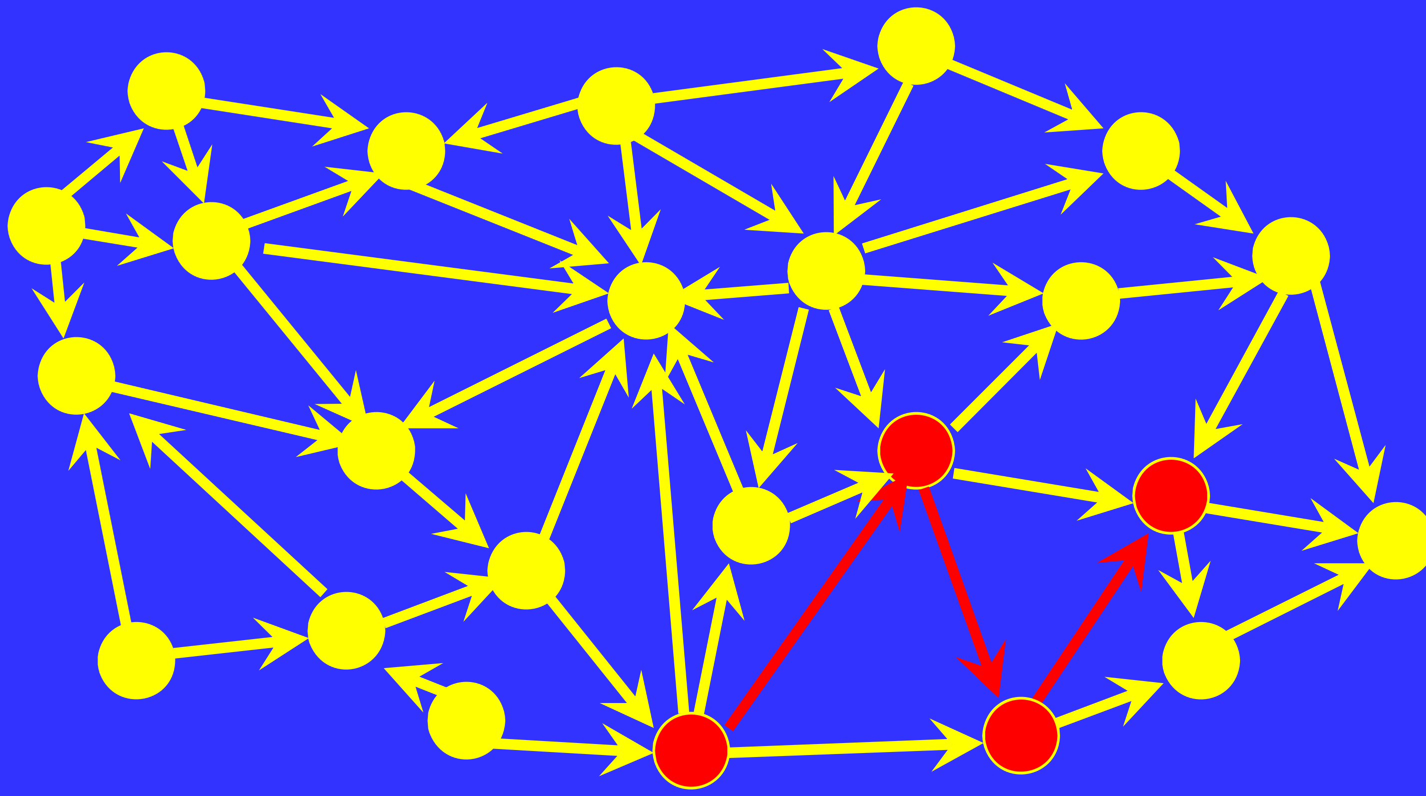

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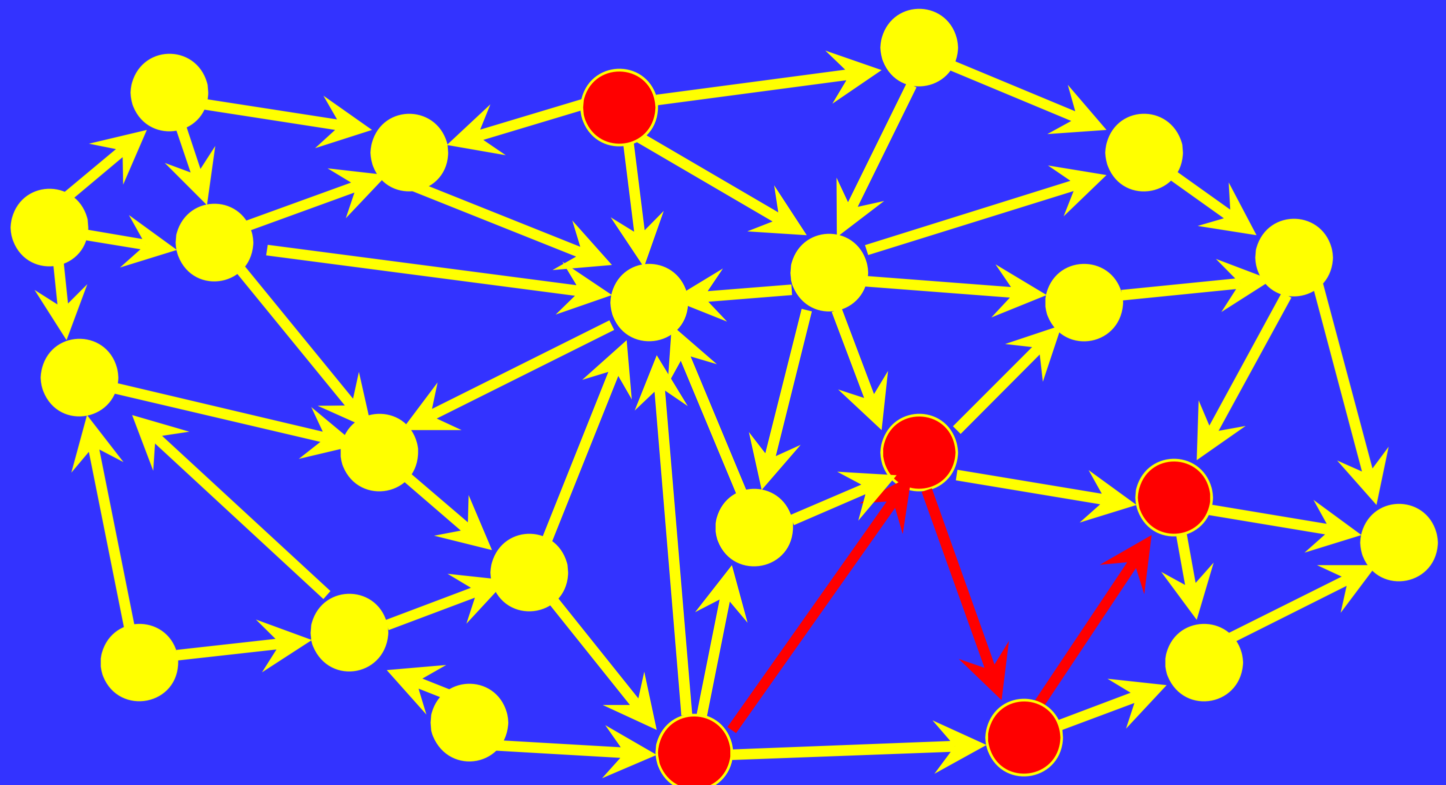
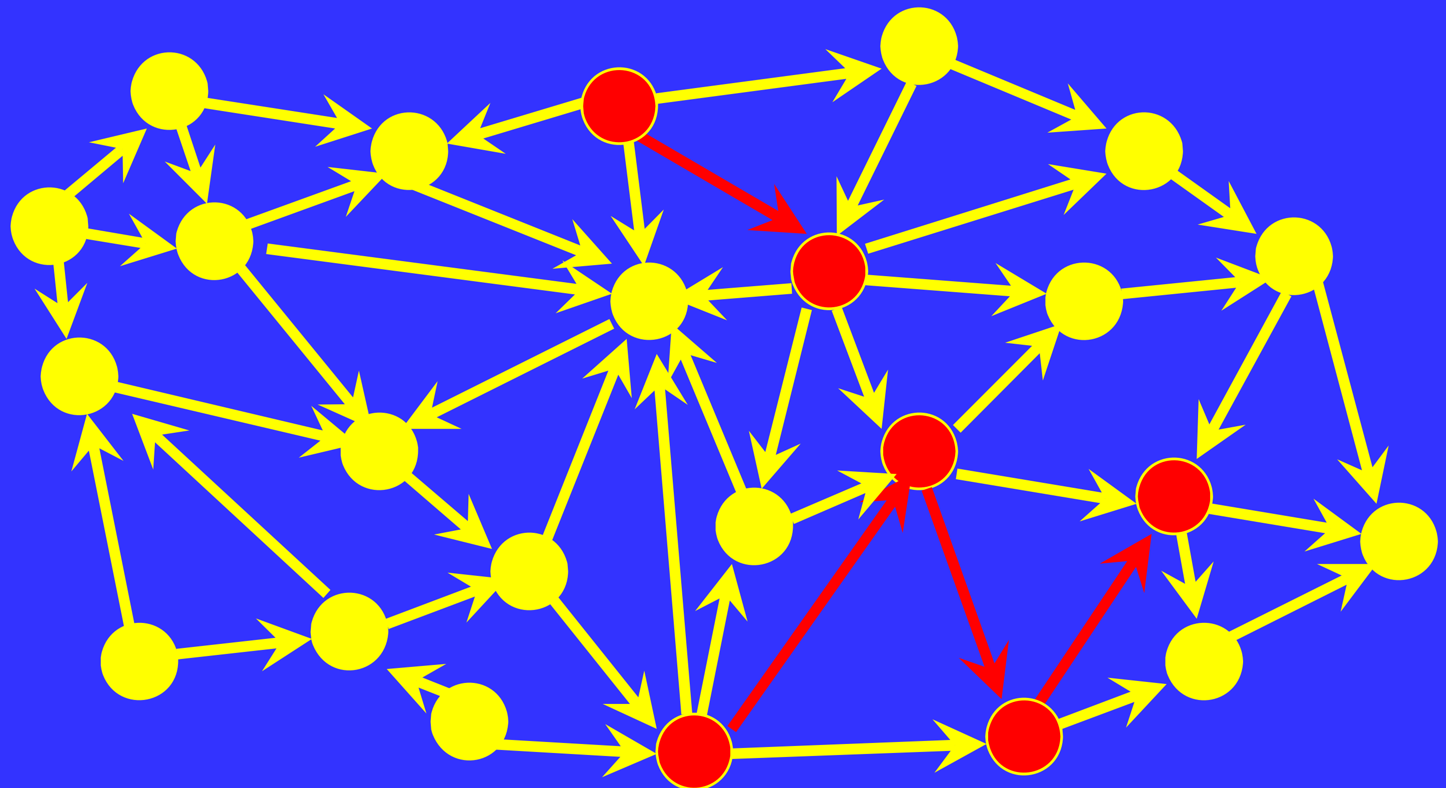
Teleportation Model

- We are "teleported" to a web page located far away



Random Surfer Model Again

- Pick another outgoing link at random



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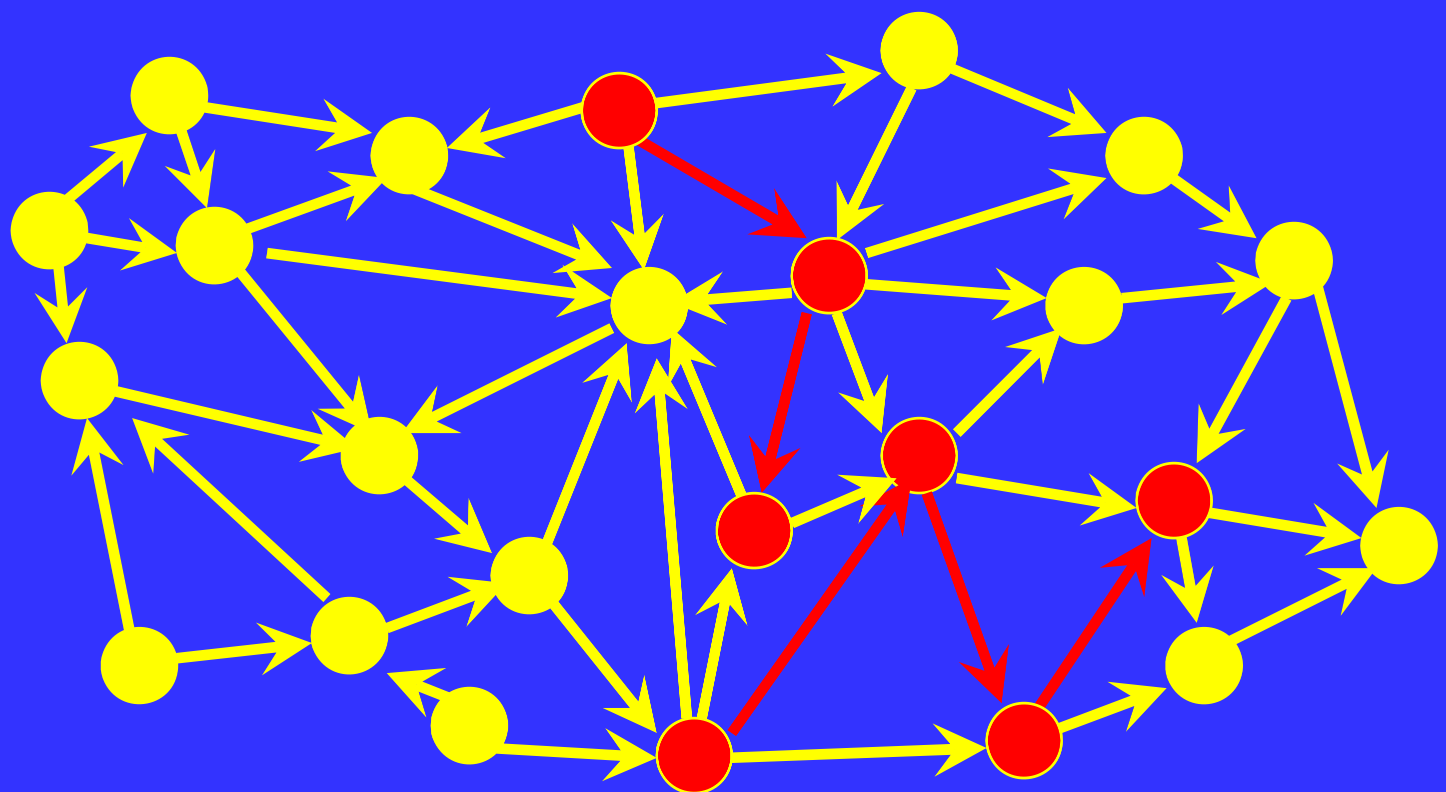
Random Surfer Model Again

- Pick another outgoing link at random



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Teleportation Model Again

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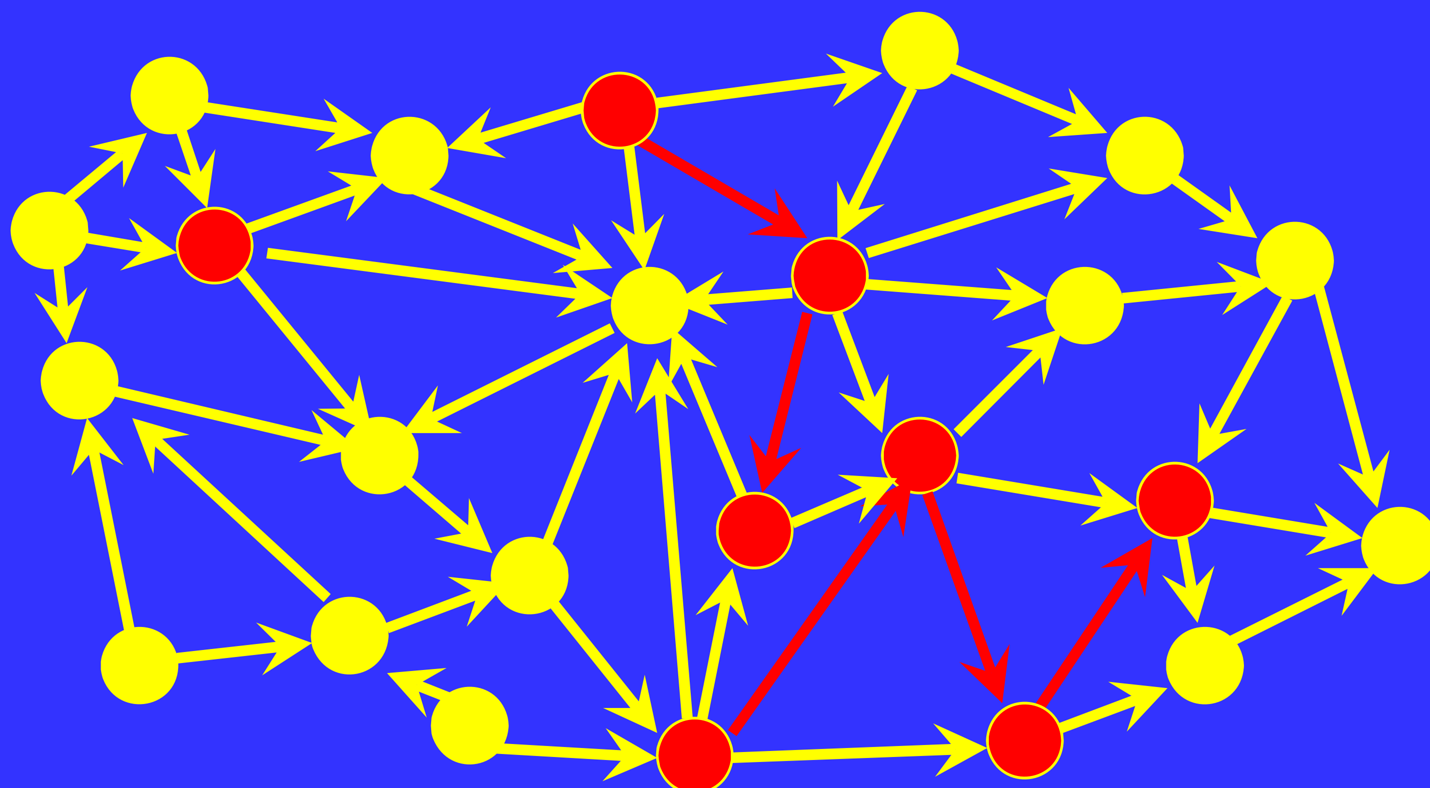

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Convex Combination of Matrices

- Teleported model is represented as a convex combination of matrices
- Instead of A we consider a matrix M defined as

$$M = (1 - m)\, A + m/n\, S \qquad m \in (0,1)$$

where S is a matrix with all entries equal to 1 and n is the number of pages

- The value $m = 0.15$ is proposed and used at Google[1]

[1] S. Brin, L. Page (1998)

Matrix M

- M is positive stochastic (convex combination of two stochastic matrices and $m \in (0,1)$)

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Matrix M and Perron Theorem

- The matrix M is primitive (M^k is positive for some k)
- M is irreducible and the corresponding graph is strongly connected (every page is connected to every page)
- The eigenvalue 1 is simple and it is the unique eigenvalue of maximum modulus
- The corresponding eigenvector is positive

PageRank Computation

PageRank Computation

- PageRank is computed with the power method

$$x(k+1) = M\, x(k)$$

- Convergence of this recursion is guaranteed by Perron Theorem because M is a primitive matrix

$$x(k) \to x^* \quad \text{for} \quad k \to \infty$$

provided that $\sum_i x_i(0) = 1$

- Remark: PageRank computation can be interpreted as finding the stationary distribution of a Markov Chain

Convergence Properties

- Asymptotic rate of convergence of power method is exponential and depends on the ratio

$$| \lambda_2(M) / \lambda_1(M) |$$

- We have

$$\lambda_1(M) = 1 \qquad \lambda_2(M) \leq 1 - m = 0.85$$

PageRank Computation with Power Method

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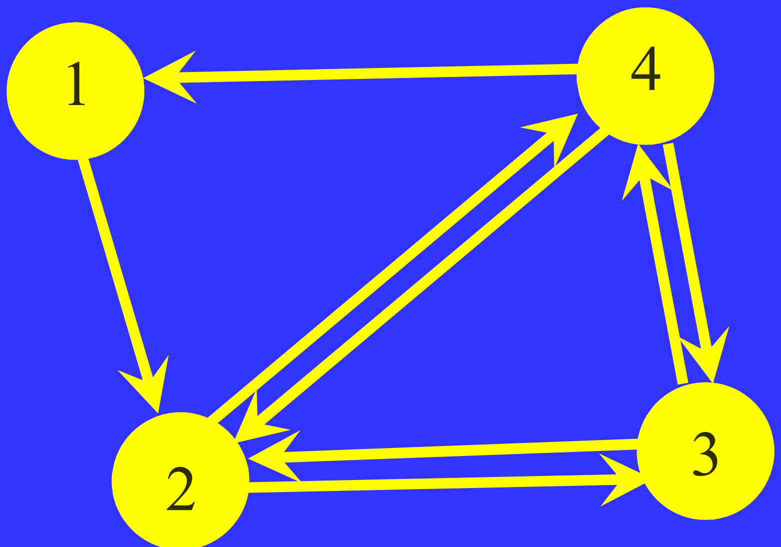

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix} \quad m = 0.15$$

$$M = \begin{bmatrix} 0.038 & 0.037 & 0.037 & 0.321 \\ 0.887 & 0.037 & 0.462 & 0.321 \\ 0.037 & 0.462 & 0.037 & 0.321 \\ 0.037 & 0.462 & 0.462 & 0.037 \end{bmatrix}$$

$$x^* = \begin{bmatrix} 0.12 & 0.33 & 0.26 & 0.29 \end{bmatrix}^T$$

Size of the Web

- The size of M is 8 billion!
- The PageRank computation requires 50-100 iterations
- This takes about a week and it is performed centrally at Google once a month
- More and more computing power is needed…

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Columbia River, The Dalles, Oregon

Randomized Algorithms

Randomized Algorithm (RA)

- Randomized Algorithm (RA): An algorithm that makes random choices during its execution to produce a result

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- Example of a "random choice" is a coin toss

heads or *tails*

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Monte Carlo and Las Vegas Randomized Algorithms

- Monte Carlo was invented by Metropolis, Ulam, von Neumann, Fermi, … (Manhattan project)
- Las Vegas first appeared in computer science in the late seventies
- Successful applications of randomized algorithms in several areas, e.g. robotics, computer science, finance, bioinformatics, communication and networking, systems and control, …

Monte Carlo Randomized Algorithm

- **Monte Carlo Randomized Algorithm:** A randomized algorithm that may produce incorrect results, but with bounded error probability

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Monte Carlo Randomized Algorithm

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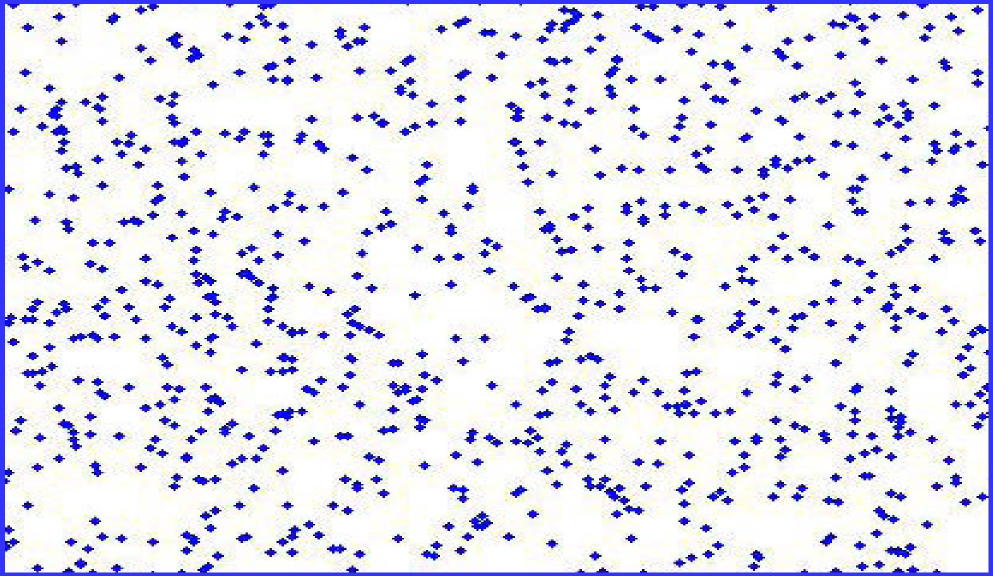
Example of Monte Carlo

- Objective is to estimate the measure of set contained in a hyperrectangle

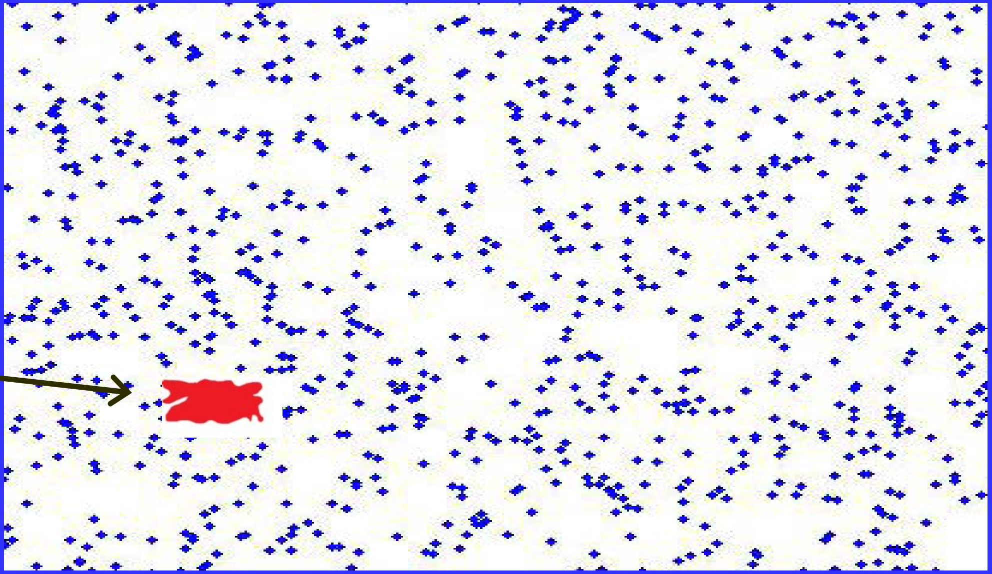
Example of Monte Carlo

- This objective can be accomplished using randomization



Example of Monte Carlo

- This objective can be accomplished using randomization
- There is a non-zero probability of error (which can be bounded) associated to randomization
- This is due to finite sample size



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Las Vegas Randomized Algorithm

- **Las Vegas Randomized Algorithm:** A randomized algorithm that always (i.e. w.p.1) produces correct results, the only variation from one run to another is the running time

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Las Vegas Randomized Algorithm

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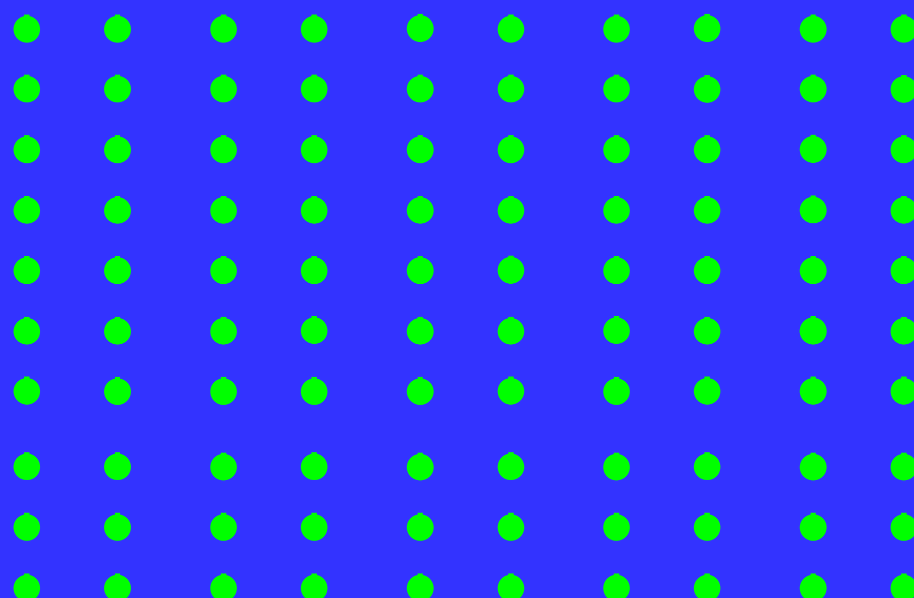
Example of Las Vegas Randomization

we consider a discrete bounding set and we explore the entire search space



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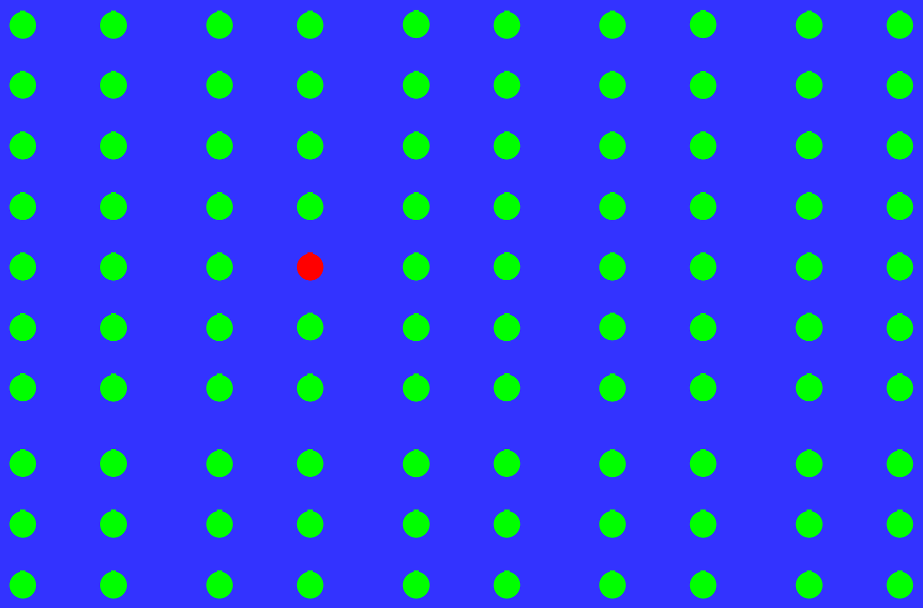

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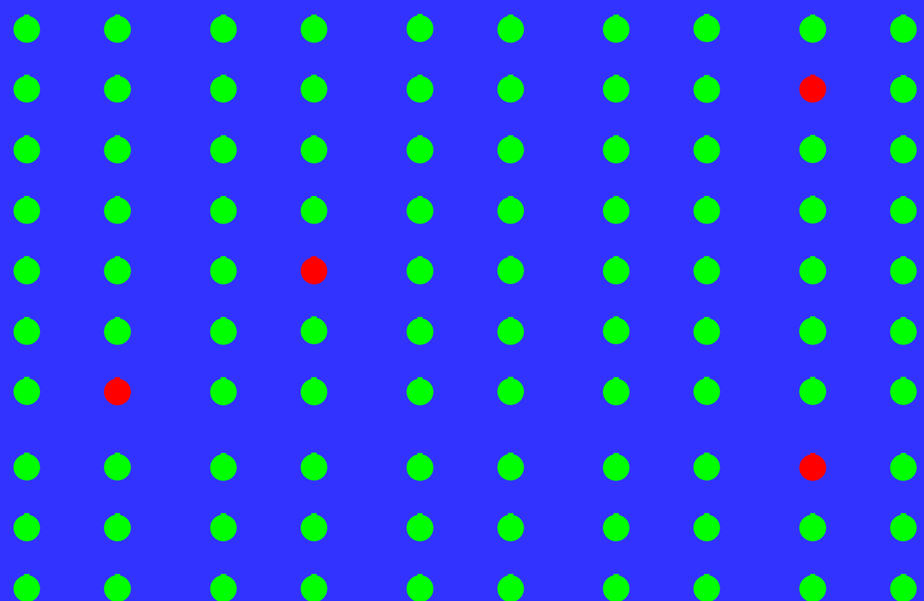

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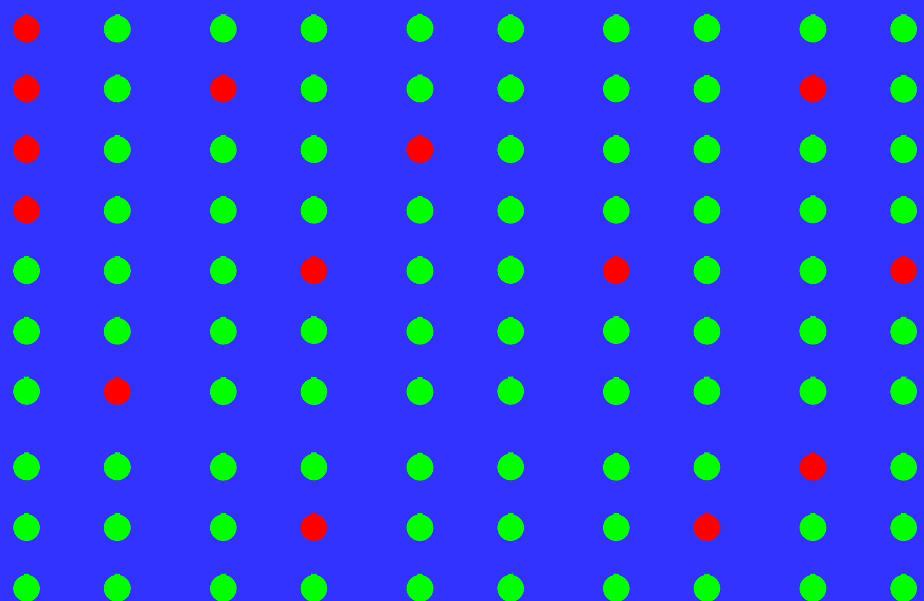

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Randomized Decentralized Algorithms for PageRank Computation

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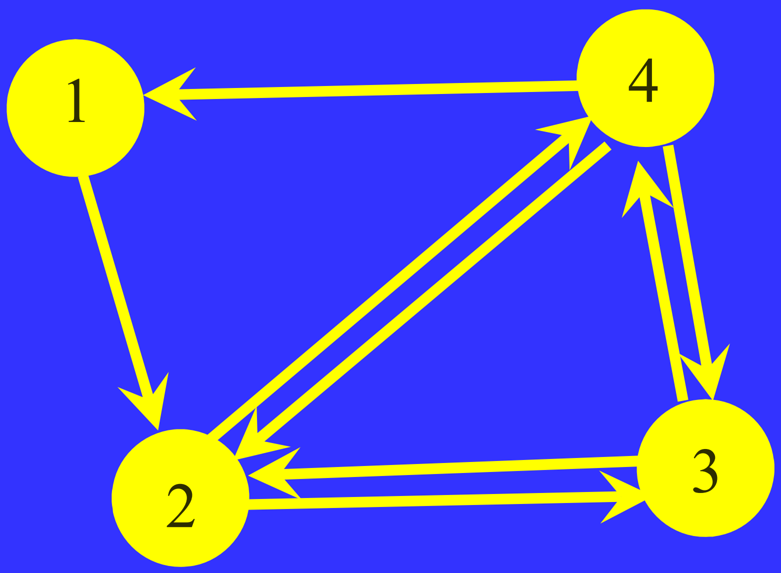
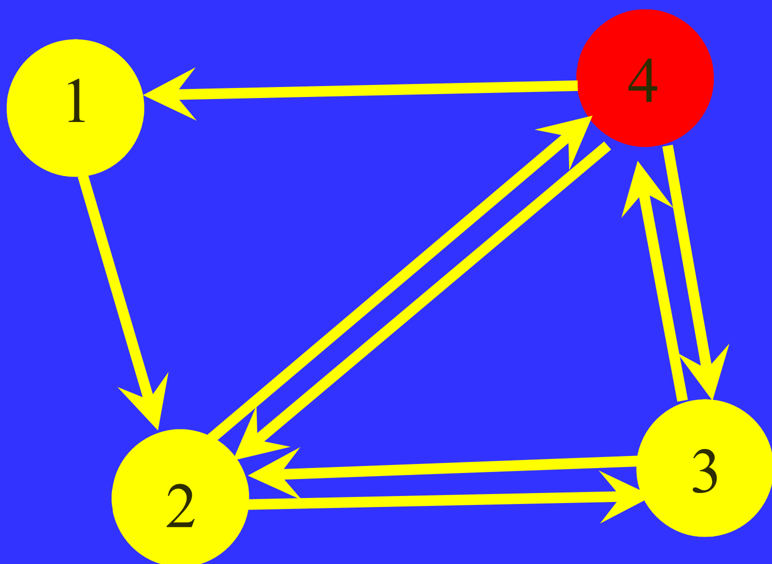
Basic Communication Protocol



Basic communication protocol: at time k the randomly selected page i initiates the PageRank update as follows:

Basic Communication Protocol



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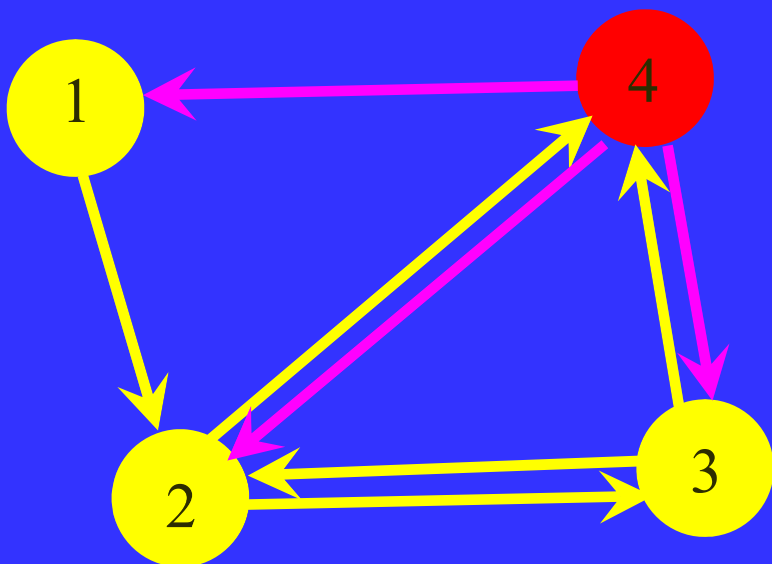

Basic communication protocol: at time k the randomly selected page i initiates the PageRank update as follows:

1. by sending the value of page i to the outgoing pages that are linked to i

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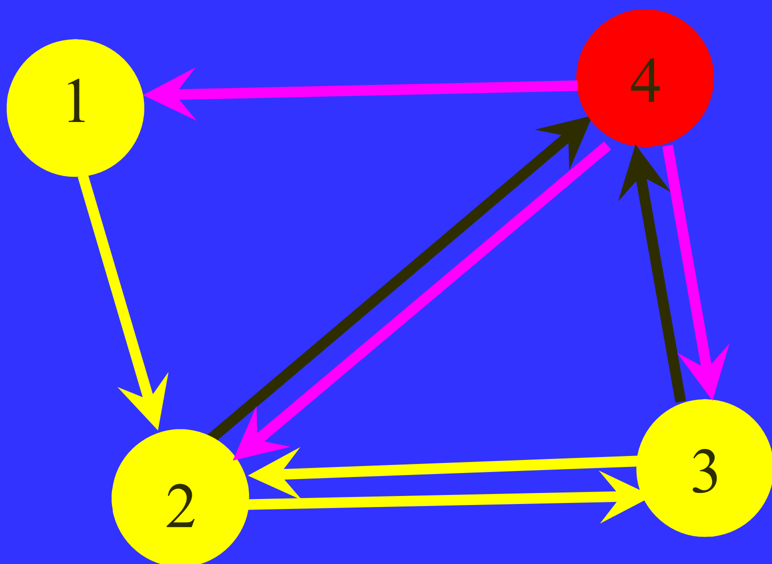

Basic communication protocol: at time k the randomly selected page i initiates the PageRank update as follows:

1. by sending the value of page i to the outgoing pages that are linked to i
2. by requesting their values from the incoming pages that are linked to page i

Las Vegas Randomized Approach

- The pages taking action are determined via a random process $\theta(k) \in \{1, \ldots, n\}$
- If at time k $\theta(k) = i$ then page i initiates PageRank update
- $\theta(k)$ is assumed to be i.i.d. with uniform probability

$$\text{Prob}\{\theta(k)=i\} = 1/n$$

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Distributed Randomized Update Scheme

- We consider the randomized update scheme

$$x(k+1) = A_{\theta(k)}\, x(k)$$

where $A_{\theta(k)}$ are the distributed link matrices (example next)

- Consider the time average

$$y(k) = 1/(k+1) \sum_i x(i)$$

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Distributed Link Matrices - 1

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} & & & 1/3 \\ & & & 1/3 \\ & & & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

Distributed Link Matrices - 2

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} & 0 & 0 & 1/3 \\ 0 & & 0 & 1/3 \\ 0 & 0 & & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

Distributed Link Matrices - 3

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

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Distributed Link Matrices - 4

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/2 & 2/3 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

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Distributed Link Matrices - 5

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2/3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 2/3 \end{bmatrix}$$

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Average Matrix $\underline{A}$

- The average matrix $\underline{A} = E[\, A_{\theta(k)} \,]$
- Since we take uniform distribution in the random process $\theta(k)$ we have

$$\underline{A} = 1/n \sum_i A_i$$

- Lemma:

(i) $$\underline{A} = 2/n\, A + (1\text{-}2/n)\, I$$

(ii) The matrices A and $\underline{A}$ have the same eigenvector corresponding to the eigenvalue 1

Modified Distributed Update Scheme

- Recall that we need to work with positive stochastic matrices
- We consider the modified distributed update scheme

$$x(k+1) = M_{\theta(k)}\ x(k)$$

where $M_{\theta(k)}$ are the modified distributed link matrices computed as

$$M_i = (1- r)\, A_i + r/n\, S \quad i = 1, 2, \ldots, n$$

and $r \in (0,1)$ is a design parameter (defined next)

Average Matrix $\underline{M}$

- The average matrix $\underline{M} = E[\, M_{\theta(k)} \,]$
- Define $r = 2m/(n - mn + 2m)$
- Lemma:

(i) $r \in (0,1)$ and $r < m = 0.15$

(ii) $\underline{M} = r/m\; M + (1 - r/m)\; I$

(iii) For $\underline{M}$ the eigenvalue 1 is simple and it is the unique eigenvalue of maximum modulus. The PageRank is the corresponding eigenvector

Theorem[1]

- Theorem:
- Using the modified distributed update scheme the PageRank is obtained through the time average y

$$E[\ \|y(k) - x^*\|^2\ ] \rightarrow 0 \quad \text{for } k \rightarrow \infty$$

provided that $\sum_i x_i(0) = 1$

- Proof: Based on the theory of ergodic matrices
- Remark: The algorithm is a LVRA

[1] H. Ishii, R. Tempo (2008)

Comments

- The average $y(k)$ can be computed recursively in terms of $y(k\text{-}1)$
- Sparsity of the matrix A_i can be preserved because

$$x(k+1) = M_i\, x(k) = (1- r)\, A_i\, x(k) + r/n\, \mathbf{1}$$

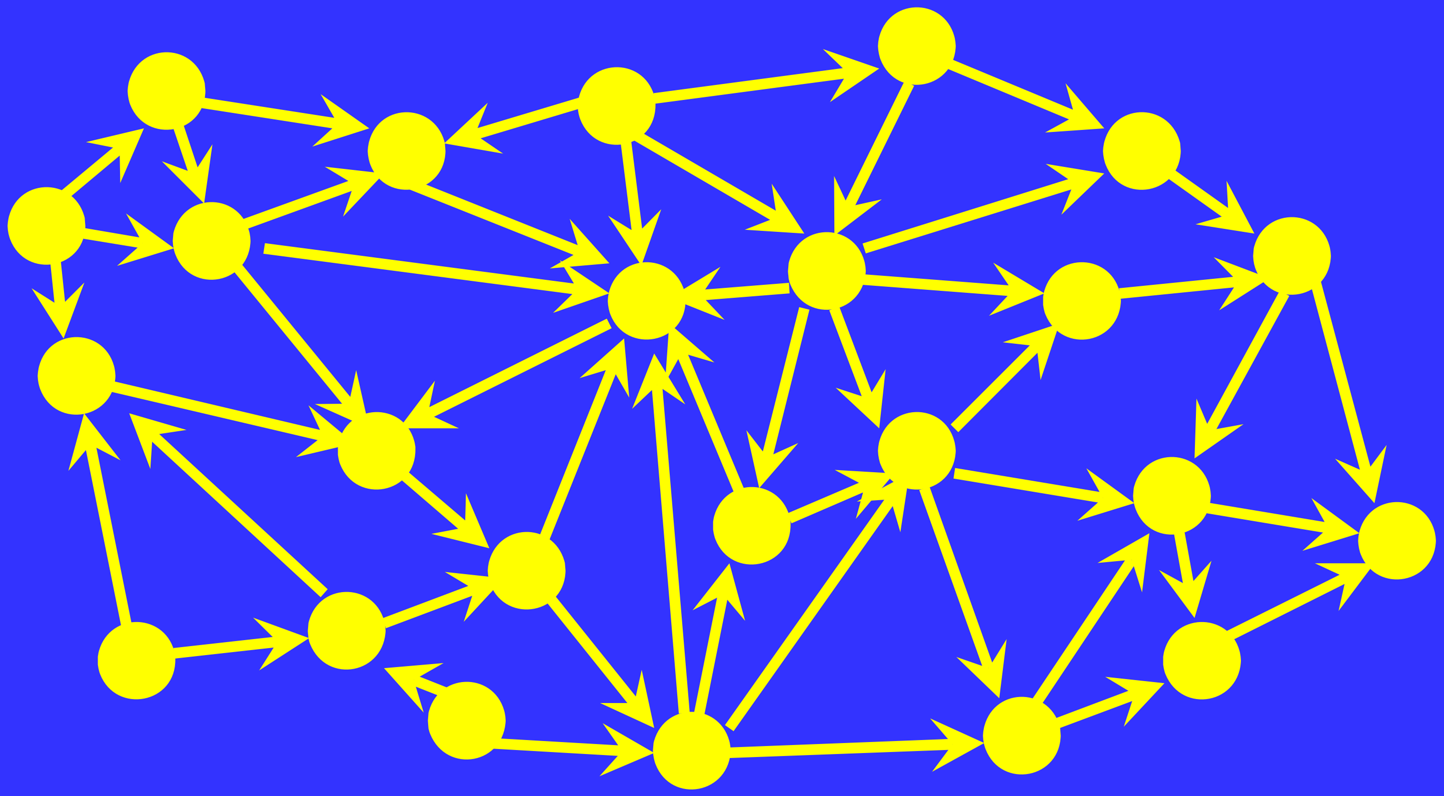
where **1** is a vector with all entries equal to one

- Convergence rate is $1/k$
- Stopping criteria to compute approximately PageRank
- Different update schemes based only on outgoing links (not incoming): Similar convergence results

Extensions: Simultaneous Updates

Recall the Random Surfer Model

- Web representation with incoming and outgoing links



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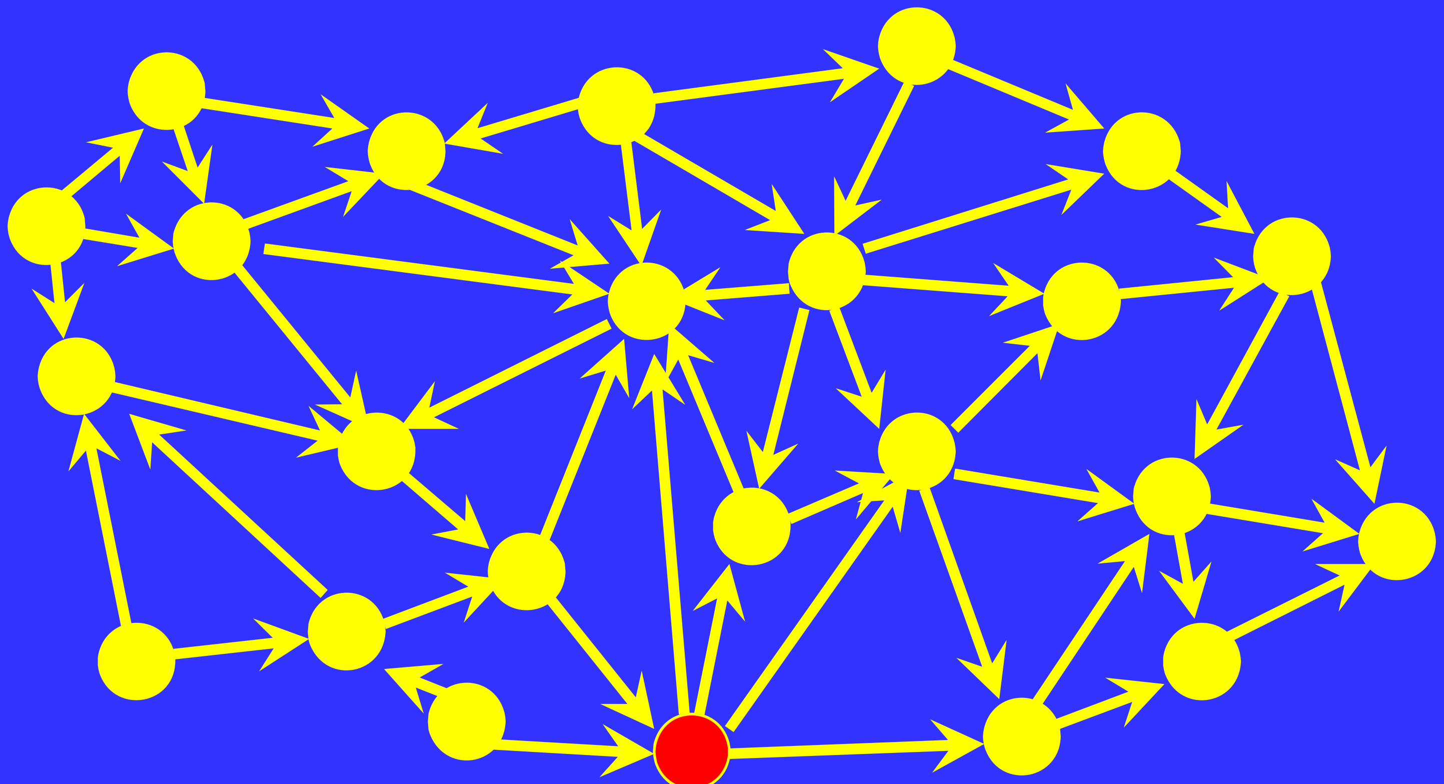
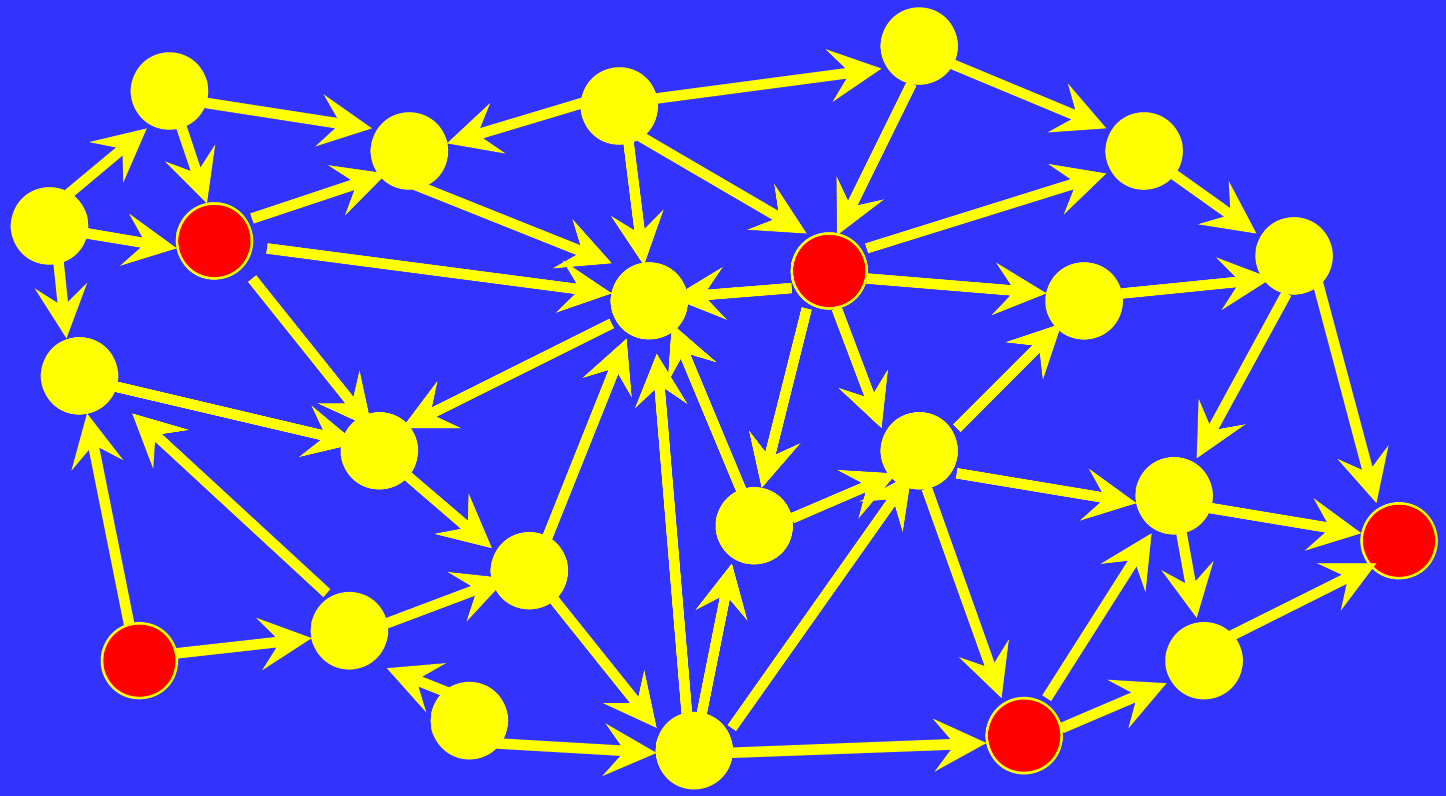
Random Update of One Page

- Random update of one page



Simultaneous Random Updates of Multiple Pages

- Simultaneous random update of multiple webpages



Convergence Results

- Simultaneous random updates of multiple pages requires defining Bernoulli processes instead of i.i.d. processes
- The math is more involved
- Similar convergence results for multiple updates can be obtained…

[1] H. Ishii, R. Tempo (2008)

Consensus and PageRank Problems

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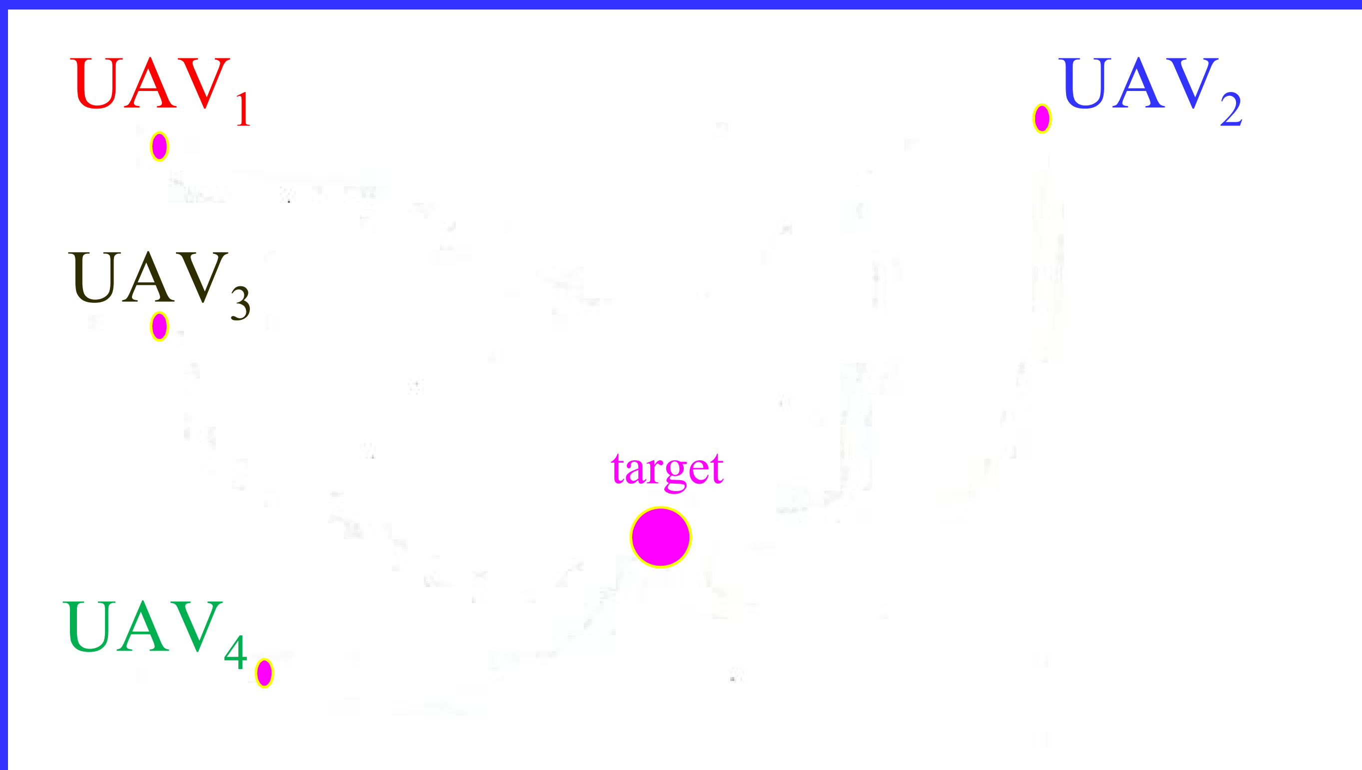
UAV

- Unmanned Aerial Vehicle (UAV) for fire detection in Sicily

- On-board equipment: various sensors, two cameras (color and infrared), GPS, ...
- DC motor
- Remote piloting and autonomous flight
- Flight endurance of about 40 min

Several UAVs (Agents) Reaching a Target



Several UAVs (Agents) Reaching a Target

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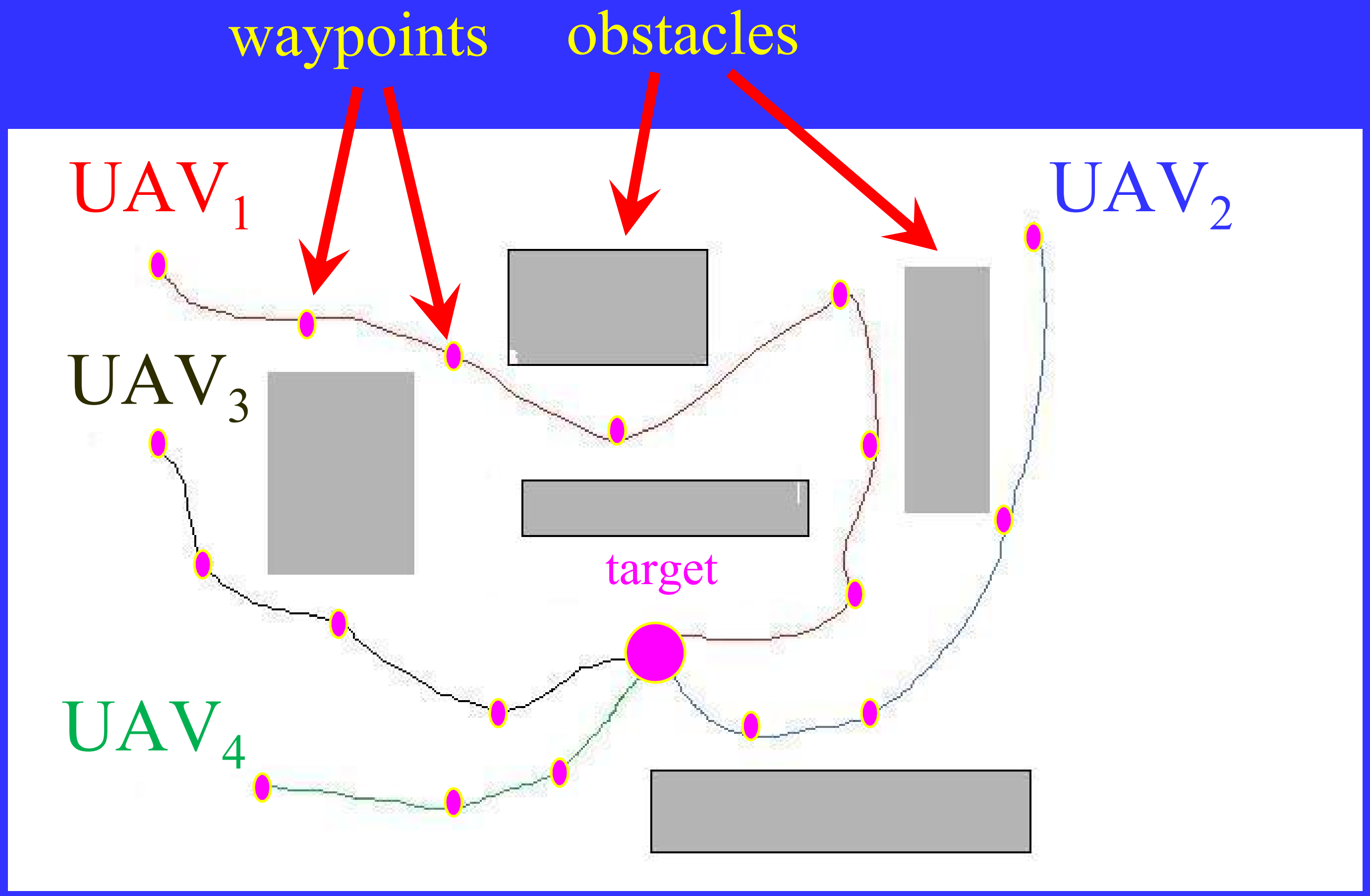

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Graph of Agents

- We consider a graph of agents which communicate using a random protocol
- The value (e.g. position) of agent i at time k is x_i
- Values of agents are updated using a random scheme

$$x(k+1) = A_{\theta(k)}\ x(k)$$

- Communication pattern (i.e. $A_{\theta(k)}$) is similar to that used for PageRank (with some technical differences)

Consensus

- We say that consensus (i.e reaching the target) is achieved if for any initial condition $x(0)$ we have

$$| x_i(k) - x_j(k) | \rightarrow 0 \quad \text{for} \quad k \rightarrow \infty$$

with probability one for all i, j

Consensus

- **Lemma:** Assume that the graph is strongly connected. Then, the scheme

$$x(k+1) = A_{\theta(k)}\ x(k)$$

achieves consensus

$$|\ x_i\,(k) - x_j(k)\ | \to 0 \quad \text{for } k \to \infty$$

with probability one for all i, j and for any initial condition $x(0)$

PageRank and Consensus

Consensus	PageRank
All agent values become equal	Page values converge to constant
Graph is strongly connected	Web is not strongly connected
Convergence w.p.1 for all x_i, x_j $\lvert x_i(k) - x_j(k) \rvert \to 0,\ k \to \infty$	MSE convergence for y $E[\ \lVert y(k) - x^* \rVert^2\ ] \to 0,\ \ k \to \infty$
Average is not necessary	Average crucial for convergence
Matrices A_i are row stochastic	Matrices M_i are column stochastic

Research Directions

- Various research directions can be explored:
  - robustness for fragile, time-varying and broken links
  - aggregation and clustering of webpages
  - web semantics

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Fragile, Time-Varying and Broken Links[1]

Page not found - connection failure

Google Toolbar

Oops! This link appears broken.

Suggestions:

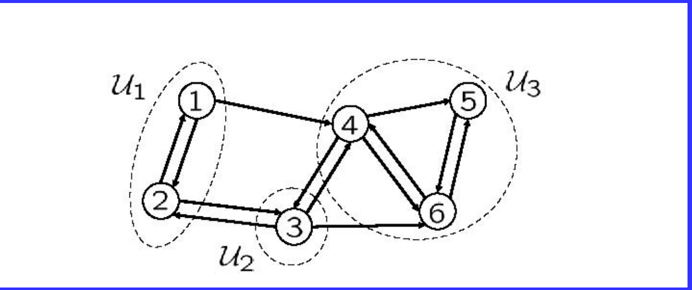
- Go to www.navy.mil
- Search on Google: [Google Search]

Google Toolbar Help - Why am I seeing this page?

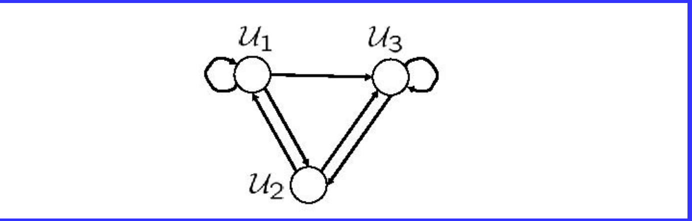
©2009 Google - Google Home

[1] H. Ishii, R. Tempo (2009)

Webpage Aggregation and Clustering[1]



original web



aggregated web

[1] H. Ishii, R. Tempo (2009)

Web Semantics

- Why the result of the web search is different if you type "roberto tempo" or "tempo roberto"?
- If you type "Boston Hotel" are you looking for an hotel in the city of Boston or are you looking for an hotel with the name "Boston"?

Why Restricting to Ranking Webpages?

- We could rank journals in order of importance…
- Eigenfactor is a good example
- Other options are possible

References and Software

- "*A Distributed Randomized Approach for the PageRank Computation*," H. Ishii and R. Tempo, IEEE Transations on Automatic Control, 2010 (to appear)
- "*Randomized Algorithms for Analysis and Control of Uncertain Systems*", R. Tempo, G. Calafiore and F. Dabbene, Springer-Verlag, 2005
- "*RACT: Randomized Algorithms Control Toolbox*"

http://staff.polito.it/roberto.tempo/

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Acknowledgment

- Acknowledgment: Research on PageRank is joint work with Hideaki Ishii