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Linearity of Algorithms and a Result of Ando

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In this paper, we point out a theorem of Ando and its application to establishing necessary and sufficient conditions for linearity of spline, interpolatory, and almost strongly optimal algorithms on L^p spaces. © 1991 Academic Press, Inc.

1. INTRODUCTION

Over the last few years, the field of *information-based complexity* has drawn much attention from researchers working in disciplines in which information about problems is partial and/or contaminated. Specific application areas include computer science, economics, control theory, and signal processing. For a detailed survey of this subject, see the recent book of Traub *et al.* (1988).

The main goal of information-based complexity is to approximate optimally a given mapping S on the linear space F (the so-called problem elements) in different settings, e.g., worst case and average case. In the

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following, we focus only on the worst case setting. Here, the problem element $f \in F$ is unknown and partial and/or contaminated information $N(f)$ about f is given. An approximation to $S(f)$ is provided by an algorithm ϕ operating on the available information $N(f)$. Within this framework, the concept of *optimal algorithm* is introduced: An optimal algorithm minimizes the maximal distance between the actual solution Sf and the estimated solution $\phi(N(f))$ for the worst problem element f in a given set, in our case the unit ball of F. A main focus of information-based complexity is to study optimality properties of several classes of algorithms, including spline, interpolatory, central and linear ones. Conditions under which spline algorithms are linear have been studied, e.g., by Packel (1986) and Werschulz and Woźniakowski (1986). More recently, Kon and Tempo (1989) have shown that given a linear space F of problem elements f, spline algorithms are linear for all linear information operators N if and only if F is a Hilbert space.

Given the above motivation, we now address the following question: *What can be said about such linearity properties for a fixed information operator*?

In this paper, we point out that a result of Ando on L^p projections can be effectively used to answer this question for the frequently occurring class of L^p spaces. For this class, it turns out that those information operators which yield linear spline, interpolatory, and almost strongly optimal algorithms can be explicitly characterized. We also use this result to prove a purely functional analytic result to the effect that an L^p space all of whose subspaces are also L^p spaces must in fact be Hilbert; that is, $p = 2$ necessarily follows from such a hypothesis.

2. Definitions and Notation

The following is a brief sketch of the definitions involved. This background is further elaborated in Traub and Woźniakowski (1980), Packel and Woźniakowski (1987), and Packel (1988).

Let F be a Banach space with F_0 the unit ball of F. Let S be a given linear operator, called the *solution operator*, mapping F into Z, with Z a Banach space. Our aim is to estimate an element Sf of Z, having only limited information about the problem element f.

Define a bounded linear operator N, called an *information operator*, which maps F into a finite dimensional linear space Y

$$N: F \to Y. \tag{2.1}$$

An *algorithm* ϕ is a transformation (in general nonlinear) from Y into Z

$$\phi: Y \to Z$$

which provides an approximation $\phi(y)$ of Sf using information $y = Nf$.

Define the *local error* $E(\phi, y)$ of an algorithm ϕ as

$$E(\phi, y) = \sup_{f' \in T(y)} \|Sf' - \phi(y)\|, \tag{2.2}$$

where $T(y) = \{f \in F_0: Nf = y\}$.

An algorithm Φ_0 is *strongly optimal* if

$$E(\Phi_0, y) = \inf_{\Phi} E(\Phi, y)$$

for all y in $N(F_0)$. Similarly, an algorithm ϕ_k is called *k-almost strongly optimal* if

$$E(\phi_k, y) \le k\, E(\phi_0, y)$$

for all y in $N(F_0)$. We say that ϕ is *almost strongly optimal* if it is *k-almost strongly optimal* for some k. An algorithm ϕ such that $\phi(y) \in S(T(y))$ for all $y \in N(F_0)$ is called *interpolatory*. A particular interpolatory algorithm is a *spline algorithm* (see Traub and Woźniakowski, 1980), defined as $\phi_s(y) = S\sigma(y)$, where the spline $\sigma(y)$ is the minimum norm element in the set $T(y)$ (when it exists), i.e.,

$$\|\sigma(y)\| = \inf_{f' \in T(y)} \|f'\|. \tag{2.3}$$

It is well known that ϕ_s is linear if F is a Hilbert space and $\ker N$ is closed.

If F is a normed linear space and G is a closed subspace, we define the projection operator $\pi_G: F \to G$ by $\pi_G f = g$ if

$$\|f - g\| = \inf_{g' \in G} \|f - g'\|.$$

Note that π_G may be only partially defined unless F is strictly convex and reflexive (which guarantees uniqueness and existence; see Singer, 1974).

3. Linearity in L^p Spaces

In this section we consider linearity of algorithms when the space of problem elements is or can be isometrically identified with an L^p space. We remark that this class includes all Hilbert spaces.

We begin with a theorem of Ando (1966).

THEOREM 1 (Ando, 1966). *Let (M, μ) be a measure space, and $F = L^p(\mu)$ for some p, $1 < p < \infty$. Let G be a closed linear subspace of F. Then the projection π_G onto G is linear if and only if the quotient space F/G is metrically isomorphic to $L^p(\nu)$, where ν is a measure on a (possibly) different space.*

With this result as our primary tool, we prove the following theorem on linearity of algorithms in L^p spaces.

THEOREM 2. *Let $F = L^p(\mu)$ be an L^p space for some $1 < p < \infty$ and μ be a measure. Let $N:F \to Y$ be a given linear bounded information operator. Then the following statements are equivalent:*

1. *$F/\ker N$ is isomorphic to an L^p space.*

For any bounded linear operator S (as in Section 2) there exists

2. *a linear interpolatory algorithm*
3. *a linear spline algorithm*
4. *a linear almost strongly optimal algorithm.*

Proof. The theorem is proved through the following sequence of implications: $1 \to 3$; $3 \to 1, 2$; $2 \to 4$; and $4 \to 3$.

That 3 implies 2 follows immediately from the general fact that spline algorithms are interpolatory (see Traub and Woźniakowski, 1980). That 2 implies 4 follows from the fact that interpolatory algorithms are 2-almost strongly optimal.

To prove that 4 implies 3, we refer the reader to the proof of the main theorem in Kon and Tempo (1989) where this implication is proved in general (see also Traub and Woźniakowski, 1980, Lemma 4.4.2).

Now we prove the equivalence of 3 and 1. First, assume that the spline algorithm $\sigma \circ S$ is linear for all S, i.e., that σ is linear. Let $K = \ker N$. Therefore, $\sigma \circ N(x)$, which maps x to the smallest element in $x + K$, is also linear. Thus, $\pi_{K+x}(0) = \sigma \circ N(x)$ is linear in x, so that for $x \in F$,

$$\pi_K(x) = \pi_{K-x}(0) + x = \sigma \circ N(-x) + x$$

is linear in x. Thus, by the theorem of Ando, F/K is isometrically isomorphic to an L^p space. The converse is proved by reversal of the above steps, completing the proof of the theorem. ∎

The following interesting operator theoretic corollary (which can also be proved directly) follows from Theorem 2.

COROLLARY. *Let F be an L^p space of dimension greater than two whose quotient modulo any closed subspace is also an L^p space. Then $p = 2$, i.e., F is a Hilbert space.*

The corollary above follows by varying N over all possible linear information operators N in Theorem 2, and using the theorem in Kon and Tempo (1989) to the effect that the linearity of spline algorithms for all information operators N implies that the problem space F is Hilbert.

Remark. This theorem also implies that the spline is *always* linear if the cardinality of information is 1, since $\mathbf{R}^1$ is always an L^p space. Hence, linear interpolatory and almost strongly optimal algorithms also always exist in this case.

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