Uncertainty and Randomization: From Theory to Applications

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The PageRank Computation in Google

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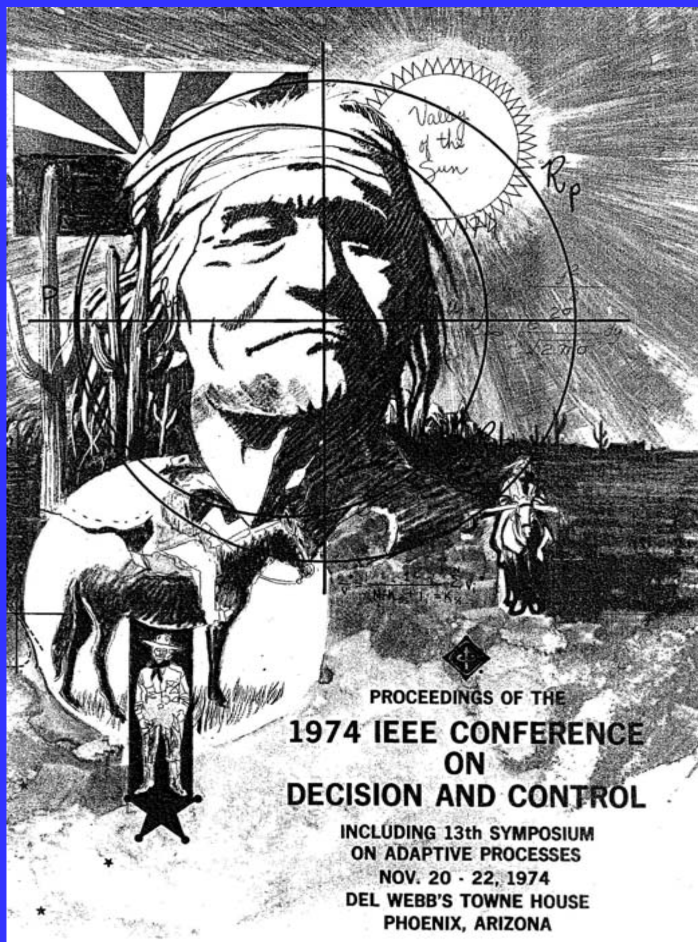
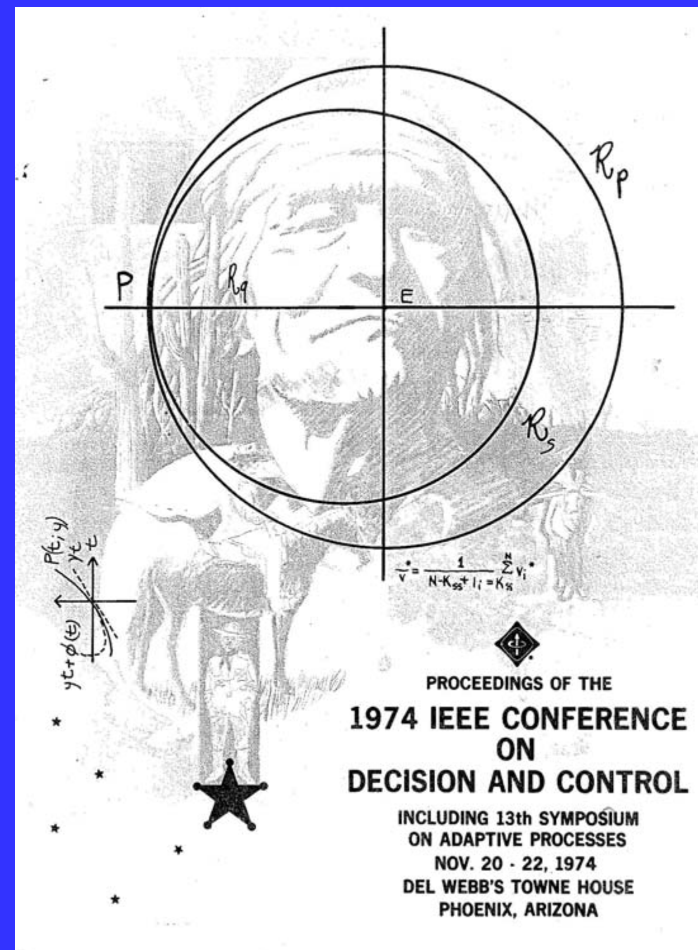
1974 IEEE CDC





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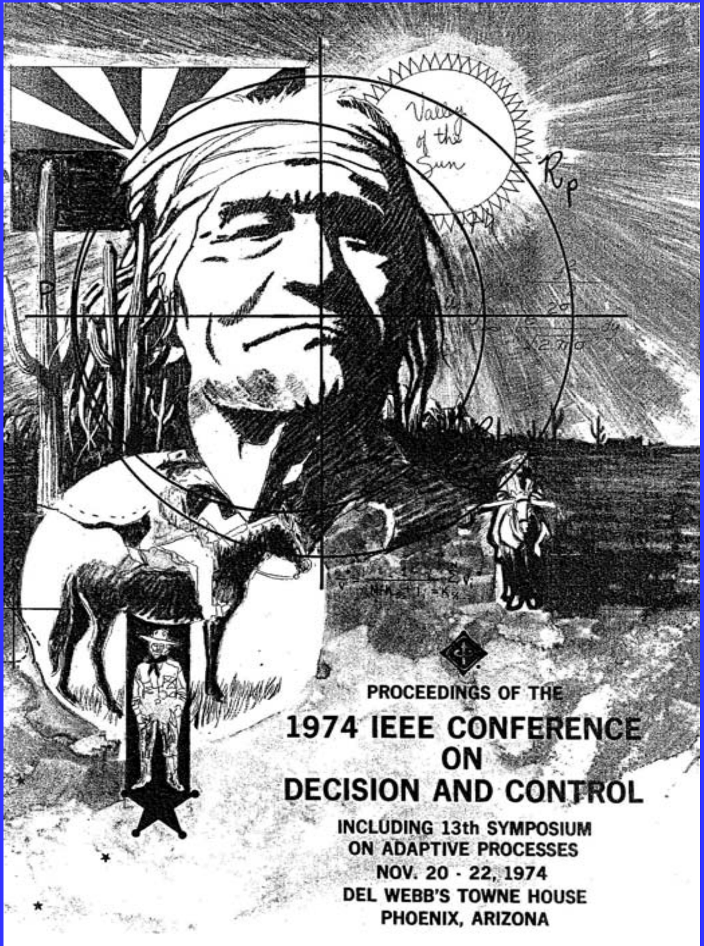
1974 IEEE CDC



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CONTROL OF NORM UNCERTAIN SYSTEMS

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Abstract

This paper is concerned with the output feedback control design of norm uncertain systems. The formulation of the uncertainty in norm is explained and its relevance to many practical cases is shown. The possibility of obtaining less uncertain outputs of the plant by means of suitable output feedback controls is investigated.

1. Introduction

The work presented here considers some aspects of the uncertain system control design in what might be called the traditional formulation of the control problem.

All signals are considered to be functions of time defined on some fixed interval (typically the interval $(-\infty, +\infty)$) and belonging to given normed spaces. The system as a whole and its building blocks are viewed as devices which accept such signals as inputs and convert them into similar signals as outputs. These devices can, but need not, be dynamical systems. If they are, no use is made of the fact. One of them, the "plant", is assumed to be partially known, that is, known to have outputs which are signals "close" to the outputs of a completely known mathematical model operating under the same conditions. Precisely the mathematical model and the plant uncertainty are defined as follows.

Let (T_1, T_2) be the considered interval of time, where T_1 and T_2 can be $-\infty$ and $+\infty$ respectively. If $T_1 > -\infty$ the plant history for $t < T_1$ is assumed fixed. Then a set U (contained in the normed space Bu) of all the admissible input signals is given. The corresponding outputs are assumed to belong to the signal normed space By and to be bounded. Then the mathematical model is expressed in the form of a function

$$\hat{y} = M(u); \qquad u \in U, \quad \hat{y} \in By$$

For any input $u \in U$ the plant output signal y is assumed to be unknown except for the fact that it belongs to the set defined by the following inequality

$$\|y - \hat{y}\| \leq E \, \|u\| + D; \quad y \in By \qquad (1)$$

where E and D are given constants not depending on the particular input chosen. Such a kind of plant uncertainty is called uncertainty in norm.

The control problem is viewed as the choice of a second device, the controller (see fig. 1.a), which together with the plant carries out at least satisfactory performance of the system as a whole.

Let G be a controller which applied to the model (fig. 1.b) meets the control problem specifications. Let $r, \tilde{u}, \tilde{y}$ be respectively the reference, the input and the output of the system in the absence of uncertainty under such a controller.

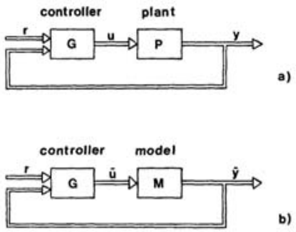

Fig. 1

35 Years Later…

- … still working on control of uncertain systems for modern applications (using different methodologies)

35 Years Later…

- … still working on control of uncertain systems for modern applications (using different methodologies)

- The PageRank Problem in Google
- Random Surfer Model and Teleportation Matrix
- A Distributed Randomized Algorithm for PageRank
- Uncertain Systems, Control and Randomization

The PageRank Problem in Google

Random Surfer Model

- Web surfer moves along randomly following the hyperlink structure
- When arriving at a page with several outgoing links, one is chosen at random, then the random surfer hyperlinks to the new one, and so on…
- The time the random surfer spends on a page is a measure of the importance of the page
- If important pages point to your page, then your page becomes important. Need to rank the pages for facilitating the web search

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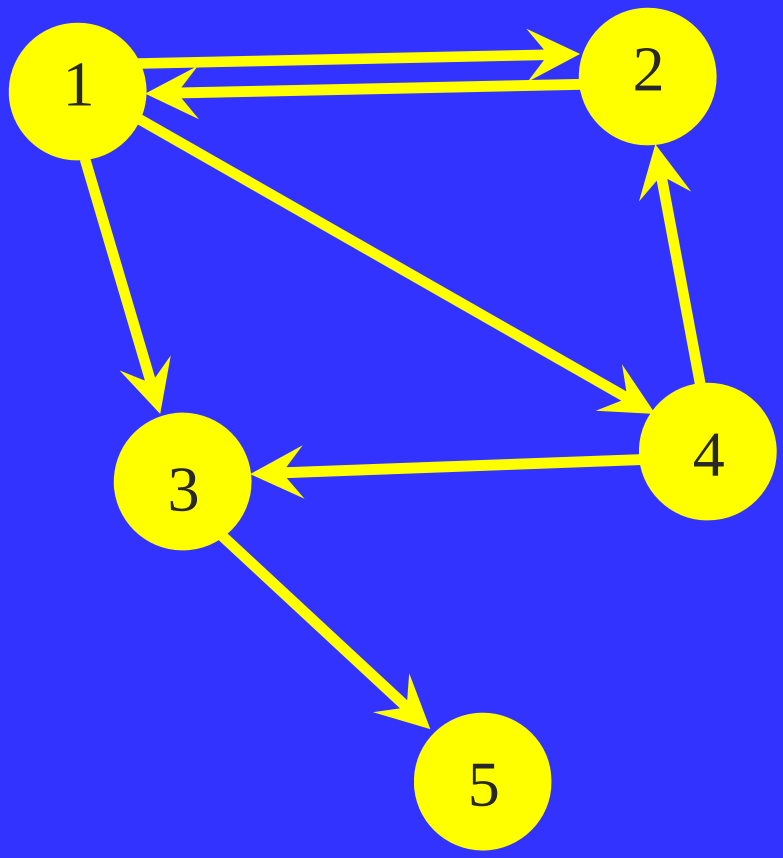
Graph Representation



- Directed graph with nodes (pages) and links representing the web
- Graph is not necessarily strongly connected
- Graph is constructed using crawlers and spiders which move continuously along the web

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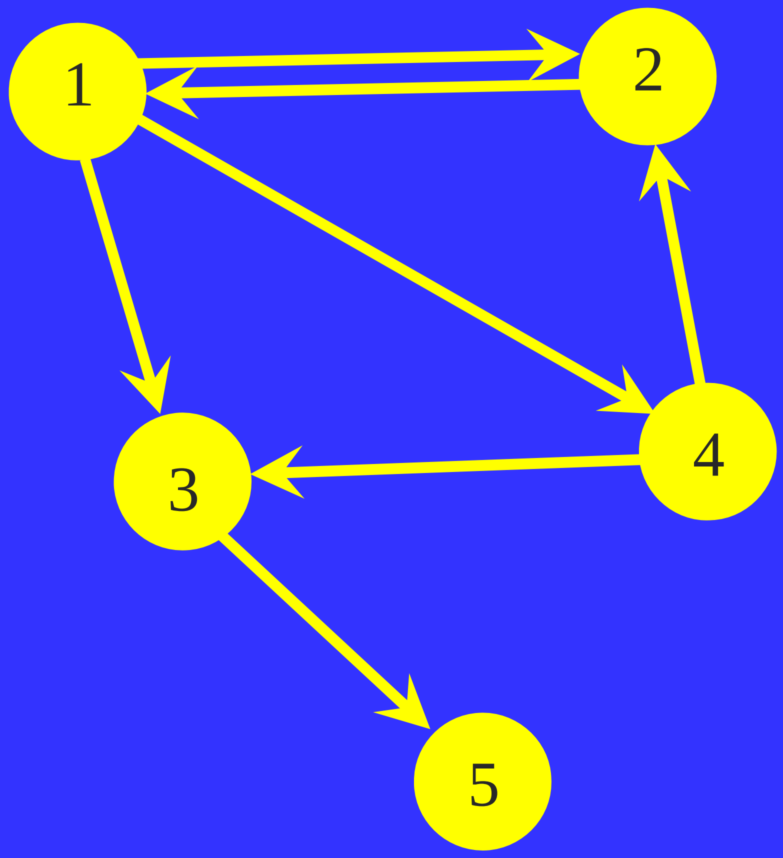
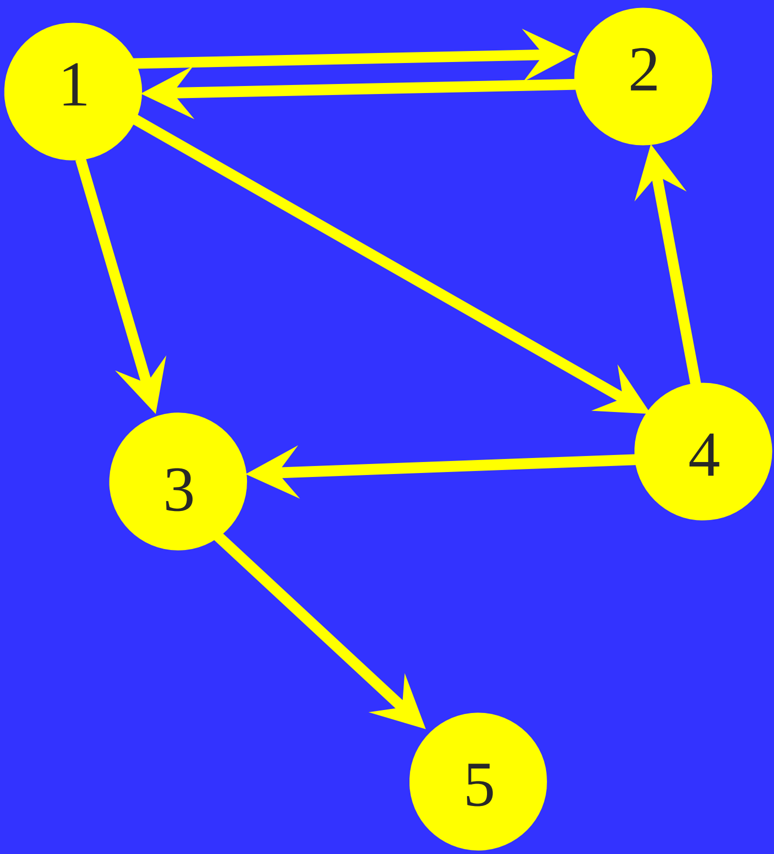
Hyperlink Matrix



- For each node we count the number of outgoing links and normalize them to 1
- Hyperlink matrix is a nonnegative (column) substochastic matrix

Hyperlink Matrix



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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PageRank: Bringing Order to the Web[1,2]

- Need to rank pages in order of importance
- The PageRank x^* is defined as

$$x^* = Ax^* \quad \text{where} \quad x^* \in [0,1]^n \text{ and } \sum_i x_i^* = 1$$

- x^* is a nonnegative unit eigenvector corresponding to the eigenvalue 1 for the hyperlink matrix A
- The question is when x^* exists and it is unique

[1] S. Brin, L. Page (1998)
[2] S. Brin, L. Page, R. Motwani, T. Winograd (1999)

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Issue of Dangling Nodes

- **First issue:** We have *dangling nodes*
- Random surfer gets “stuck” when visiting a pdf file
- In this case the “back button” of the browser is used
- Mathematically, the hyperlink matrix is nonnegative and (column) substochastic
- **Easy fix:** Add artificial links to make the matrix stochastic

Easy Fix

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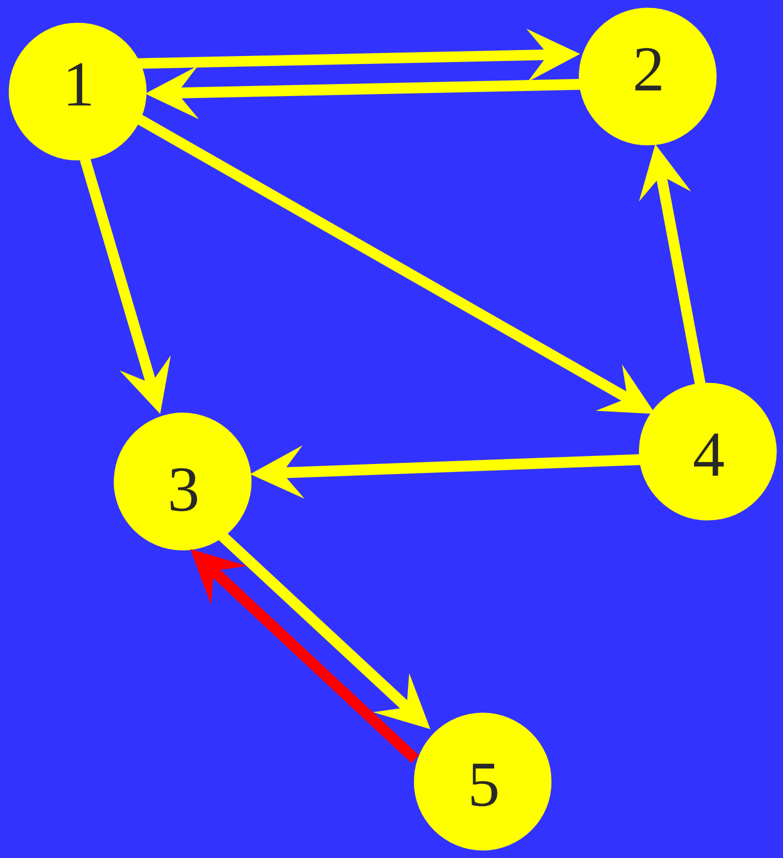
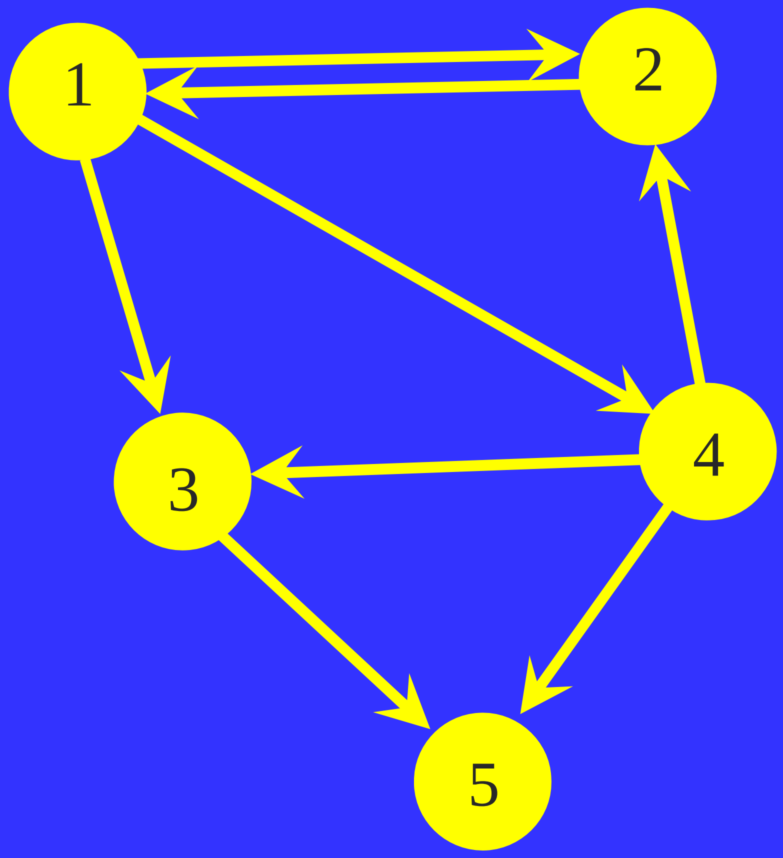
- Page 5 is a dangling node
- We add an outgoing link



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/2 & 1 \\ 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

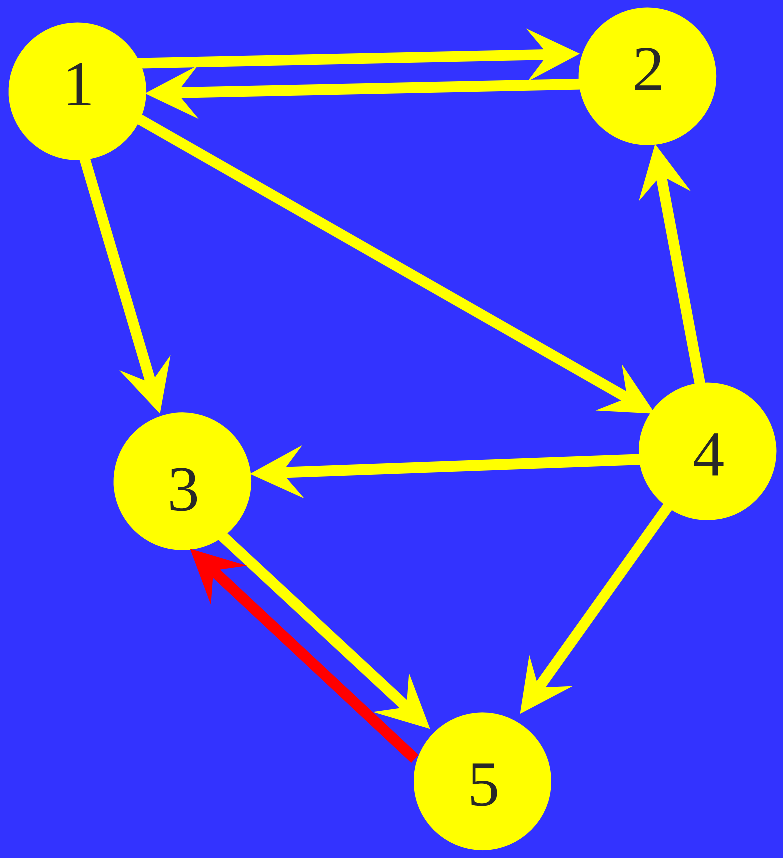
But in General the Fix is not so Easy…



- Page 5 has two incoming links

But in General the Fix is not so Easy…

- We add an outgoing link from 5 to 3…



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1 \\ 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \end{bmatrix}$$

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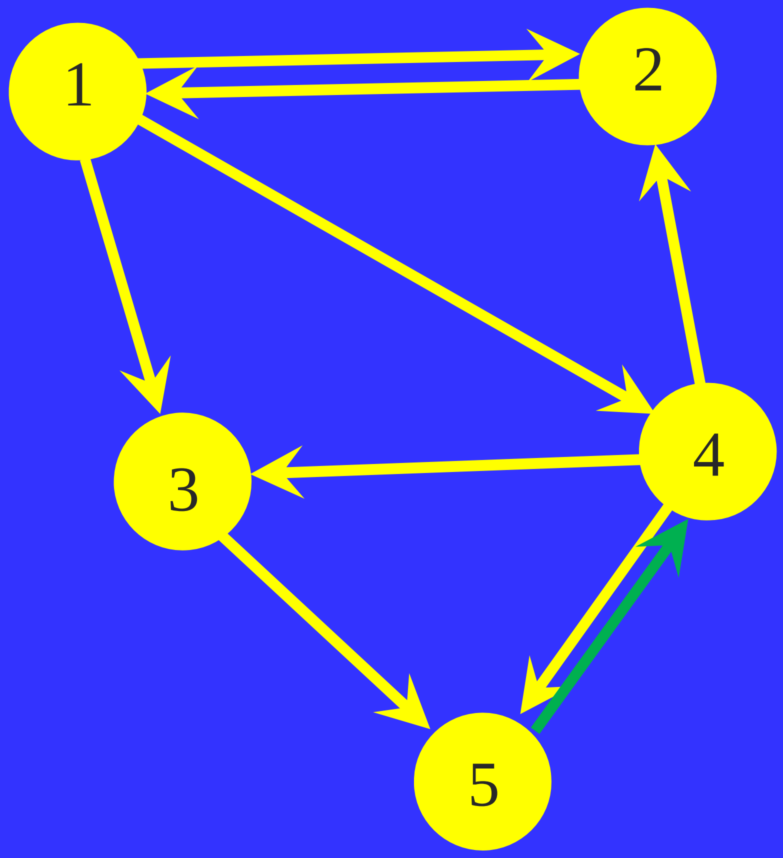
But in General the Fix is not so Easy…

- … or we add an outgoing link from 5 to 4?



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 \\ 1/3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/3 & 0 \end{bmatrix}$$

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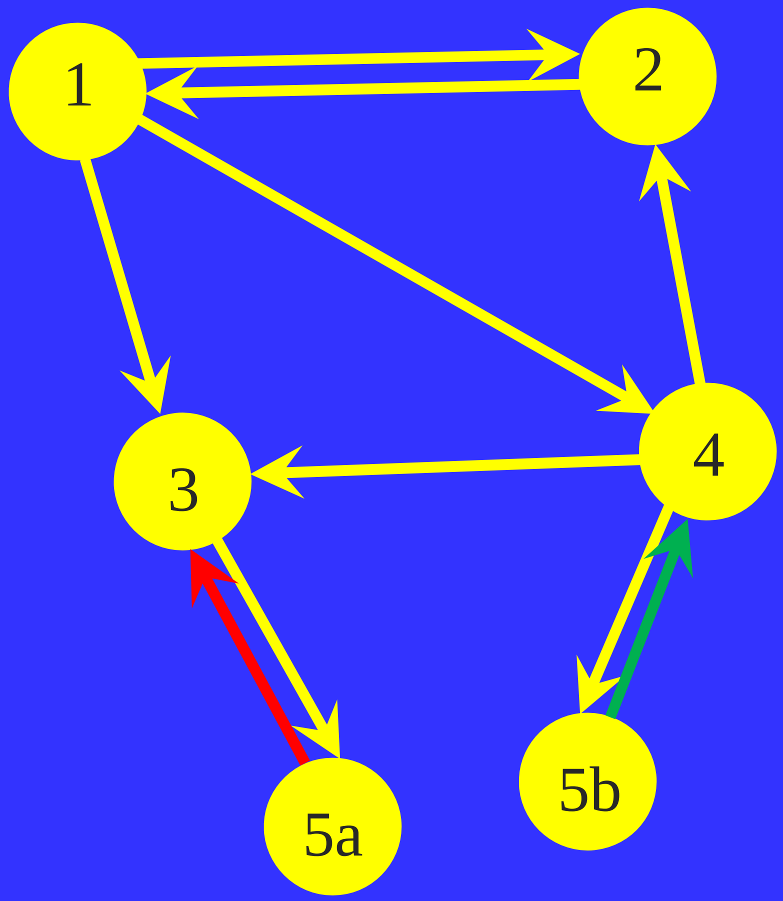
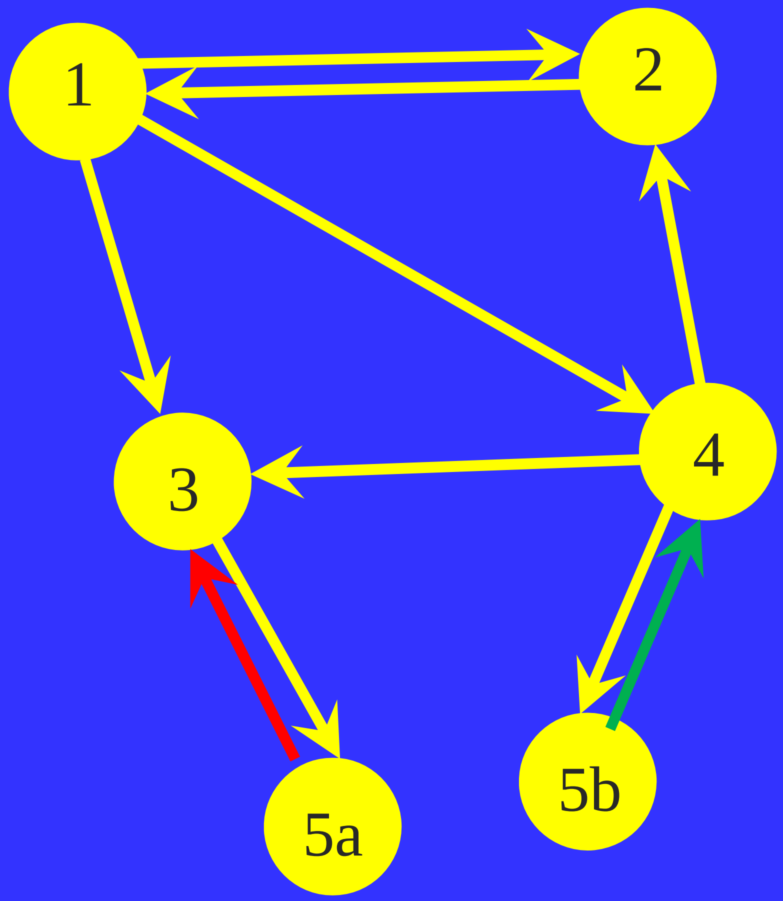
Modified Hyperlink Matrix



- A solution may be to break page 5 into two pages 5a and 5b
- This artificially changes the number of pages (not only the number of links) and the topology of the network

Modified Hyperlink Matrix



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \end{bmatrix}$$

Assumption: No Dangling Nodes

- We assume that there are no dangling nodes
- This implies that A is a nonnegative stochastic matrix (instead of substochastic) having at least one eigenvalue equal to one
- This eigenvalue is not necessarily unique

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Teleportation Matrix

- **Second issue:** The random surfer may get bored after a while, and decides to "jump" to another page not directly connected to that currently visited
- Instead of *A* we consider a matrix *M* defined as

$$M = (1 - m)\, A + m/n\, S \qquad m \in (0,1)$$

where *S* is a matrix with all entries equal to 1 and *n* is the number of pages

- The value $m = 0.15$ is proposed and used at Google[1]

[1] S. Brin, L. Page (1998)

Matrix *M* and Perron-Frobenius Theorem

- M is positive stochastic (convex combination of two stochastic matrices and $m \in (0,1)$)
- M is irreducible and the corresponding graph is strongly connected (every page is directly connected to every page)
- M is primitive because M^k is positive for some k ($k = 1$)
- **Perron-Frobenius Theorem:** For a positive stochastic matrix M there exists a unique positive eigenvector for the eigenvalue 1

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PageRank Computation

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PageRank Computation

- PageRank is computed with the power method

$$x(k+1) = M\, x(k)$$

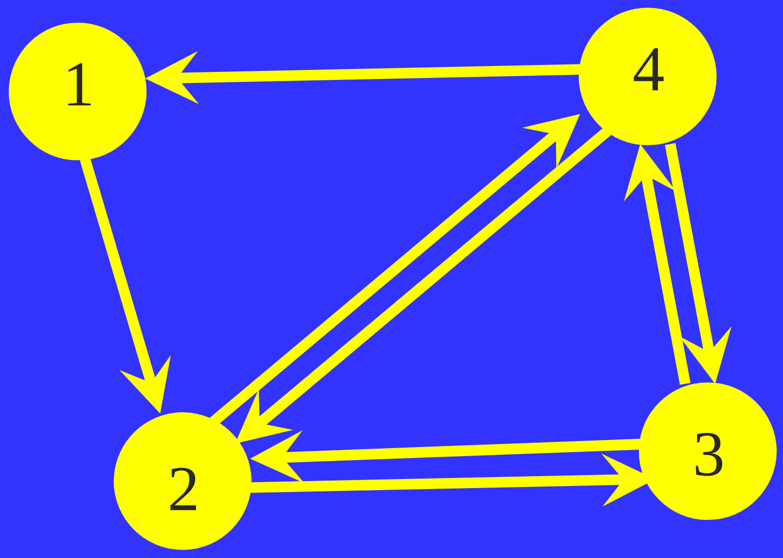
- Convergence of this recursion is guaranteed by Perron-Frobenius Theorem for any initial condition $x(0)$ because M is a positive stochastic matrix

$$x(k) \rightarrow x^* \quad \text{for} \quad k \rightarrow \infty$$

provided that $\sum_i x_i(0) = 1$

- **Remark:** PageRank computation can be interpreted as finding the stationary point of a Markov Chain

PageRank Computation with Power Method



$$A = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix} \qquad m = 0.15$$

$$M = \begin{bmatrix} 0.038 & 0.037 & 0.037 & 0.321 \\ 0.887 & 0.037 & 0.462 & 0.321 \\ 0.037 & 0.462 & 0.037 & 0.321 \\ 0.037 & 0.462 & 0.462 & 0.037 \end{bmatrix}$$

$$x^* = \begin{bmatrix} 0.12 & 0.33 & 0.26 & 0.29 \end{bmatrix}^T$$

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Size of the Web

- The size of *M* is 8 billion!
- The PageRank computation requires 50-100 iterations
- This takes about a week and it is performed centrally at Google once a month
- We need more efficient methods of computations…

Randomized Decentralized Approach

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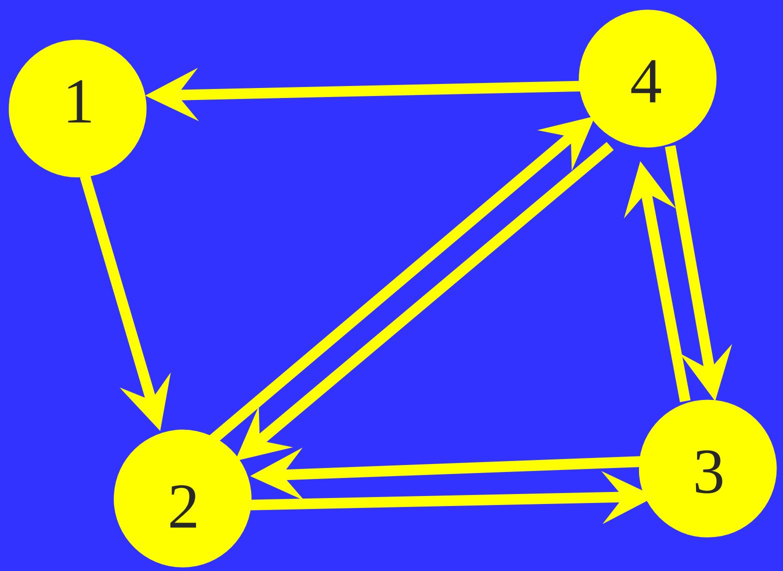
Randomized Decentralized Approach

- **Main idea:** Develop a decentralized approach for computing PageRank (instead of a centralized approach which involves the entire web)

Randomized Decentralized Approach

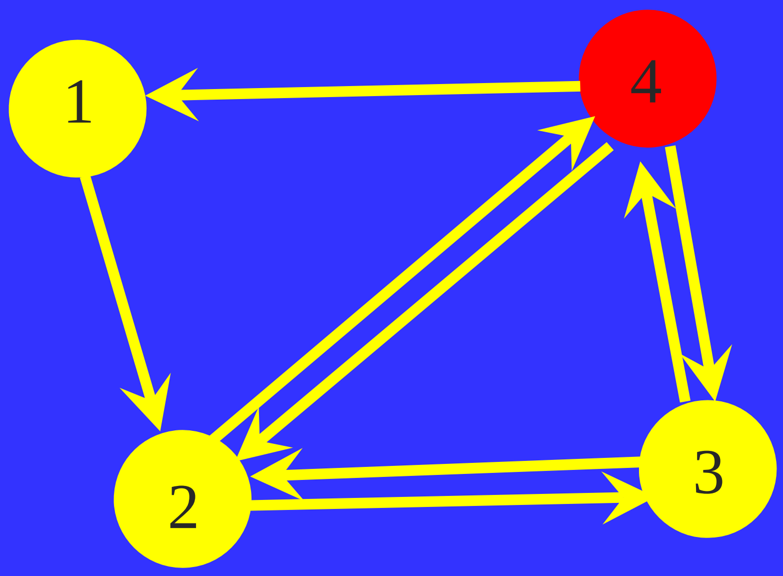
- **Main idea:** Develop a decentralized approach for computing PageRank (instead of a centralized approach which involves the entire web)
- Approach is randomization-based (Las Vegas type)

Basic Communication Protocol



Basic communication protocol: at time *k* the randomly selected page *i* initiates the PageRank update as follows:

Basic Communication Protocol



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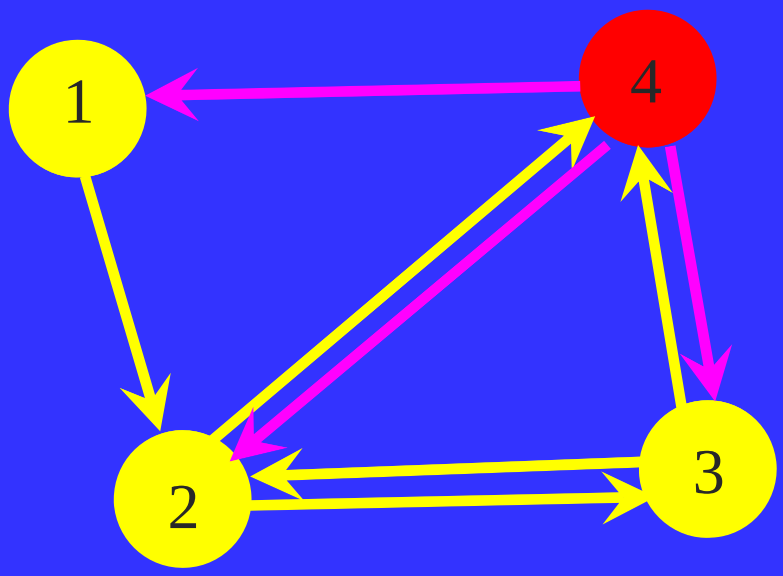

Basic communication protocol: at time *k* the randomly selected page *i* initiates the PageRank update as follows:

1. by sending the value of page *i* to the outgoing pages that are linked to *i*

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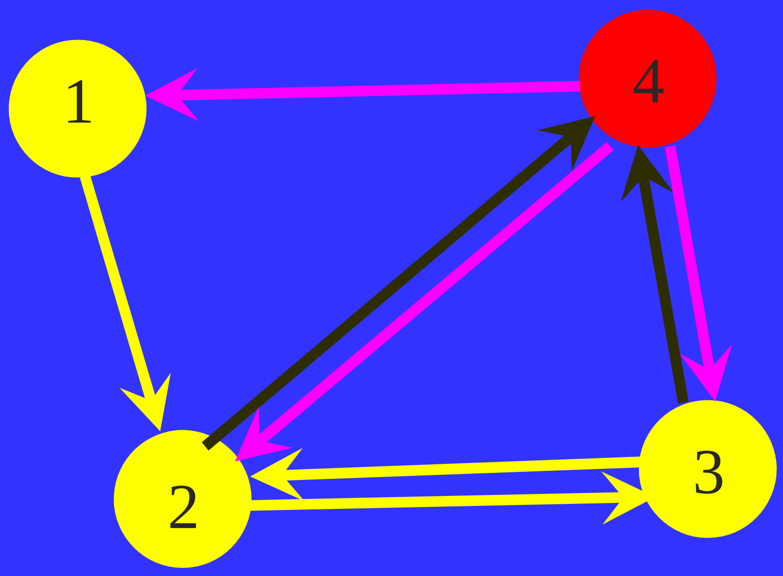
Basic Communication Protocol



Basic communication protocol: at time *k* the randomly selected page *i* initiates the PageRank update as follows:

1. by sending the value of page *i* to the outgoing pages that are linked to *i*
2. by requesting their values from the incoming pages that are linked to page *i*

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Las Vegas Randomized Approach

- The pages taking action are determined via a random process $\theta(k)$
- At time k page i initiates PageRank update with uniform probability

$$\text{Prob}\{\theta(k)=i\} = 1/n$$

Distributed Randomized Update Scheme

- We consider the randomized update scheme

$$x(k+1) = A_{\theta(k)}\, x(k)$$

where $A_{\theta(k)}$ are the distributed link matrices (example next)

- Consider the time average

$$y(k) = 1/(k+1) \sum_i x(i)$$

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Distributed Link Matrices - 1

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} & & & 1/3 \\ & & & 1/3 \\ & & & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

Distributed Link Matrices - 2

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} & 0 & 0 & 1/3 \\ 0 & & 0 & 1/3 \\ 0 & 0 & & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

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Distributed Link Matrices - 3

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

Distributed Link Matrices - 4

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/2 & 2/3 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

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Distributed Link Matrices - 5

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2/3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1/2 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 2/3 \end{bmatrix}$$

Modified Distributed Update Scheme

- Recall that we need to work with positive stochastic matrices
- We consider the modified distributed update scheme

$$x(k+1) = M_{\theta(k)}\; x(k)$$

where $M_{\theta(k)}$ are the modified distributed link matrices computed as

$$M_i = (1- r)\, A_i + r/n\, S \quad i = 1, 2, \ldots, n$$

and $r \in (0,1)$ is a design parameter

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Theorem[1]

- **Theorem:** Take $r = 2m/(n - mn + 2m) \in (0,1)$
- Using the modified distributed update scheme the PageRank is obtained through the time average y

$$E[\, \|y(k) - x^*\|^2 \,] \to 0 \quad \text{for } k \to \infty$$

provided that $\sum_i x_i(0) = 1$

- **Proof:** Based on the theory of ergodic matrices
- **Remark:** The algorithm is a LVRA

[1] H. Ishii, R. Tempo (2008)

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Technical Comments

- The average $y(k)$ can be computed recursively in terms of $y(k\text{-}1)$
- Sparsity of the matrix A_i can be preserved because

$$x(k+1) = M_i\, x(k) = (1\text{-}\ r)\, A_i\, x(k) + r/n\, \mathbf{1}$$

where **1** is a vector with all entries equal to one

- Recursion $x(k+1) = (1\text{-}\ r)\, A_1\, x(k)$ can be carried on as

$$x_3(k+1) = (1\text{-}\ r)\, x_3(k)$$

$$z(k+1) = (1\text{-}\ r)\, B\, z(k) \quad z=[x_1, x_2, x_4]^{\mathrm{T}}$$

- Convergence rate is $1/k$

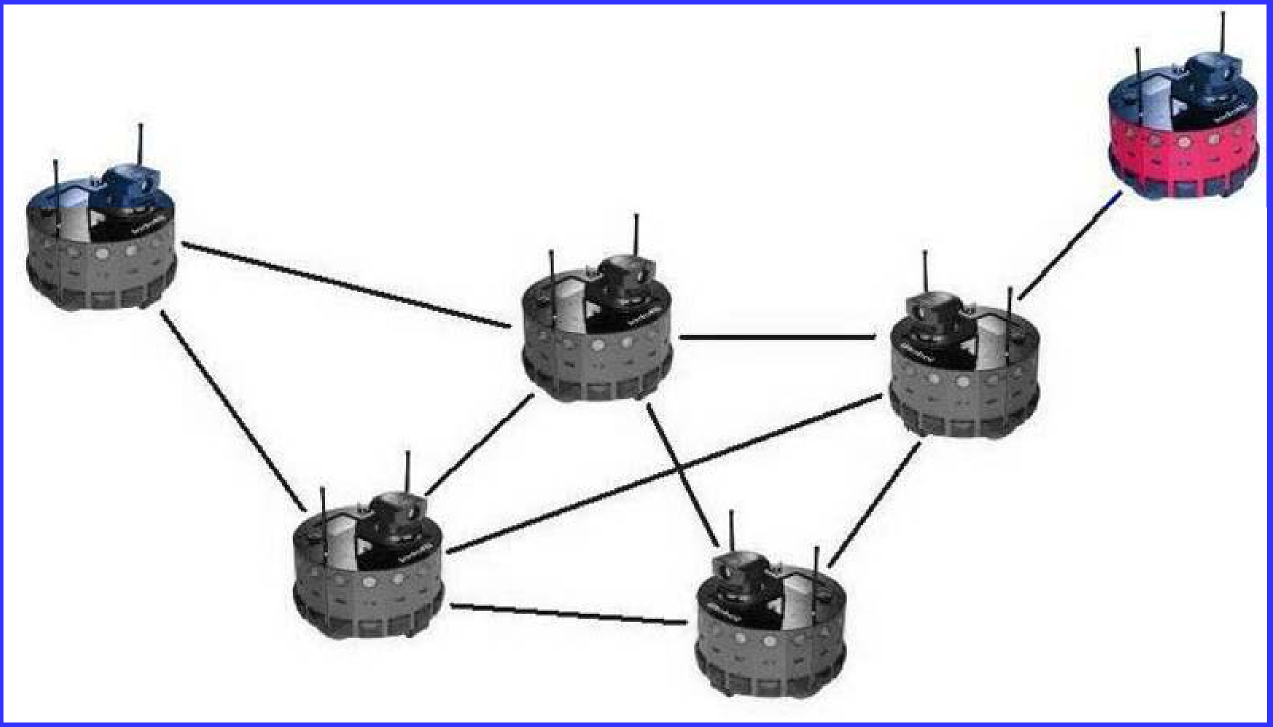
More Deeply into Distributed Randomized Schemes

- Different update schemes based only on outgoing links (not incoming)
- Simultaneous updates of multiple webpages
- Stopping criteria to compute approximately PageRank
- Improve convergence rate (power method is exponential)

Uncertain Systems, Control, Randomization and Google

PageRank and Control

- Consensus control of multi-agent systems (e.g. robots)



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PageRank and Consensus

- For consensus problems (stochastic version) we consider a graph representing a network of agents x_i

Consensus	PageRank
All agent values become equal	Page values converge to constant
Graph is strongly connected	Web is not strongly connected
Convergence w.p.1 for all x_i, x_j $\lvert x_i(k) - x_j(k)\rvert \to 0,\ k \to \infty$	MSE convergence for y $E[\ \lVert y(k) - x^*\rVert^2\ ] \to 0,\ k \to \infty$
Matrices A_i are row stochastic	Matrices A_i are column stochastic

PageRank and Uncertain Systems

- PageRank computation in the presence of uncertain, time-varying and broken links (LP solution)

Acknowledgment

- **Acknowledgment:** Research on PageRank is joint work with Hideaki Ishii
- "*Randomized Algorithms for Analysis and Control of Uncertain Systems*" by RT et al, 2005

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