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This book arises from slides and lecture notes that I have used over the years in my courses Financial Markets and Instruments and Financial Engineering, which were offered at Politecnico di Torino to graduate students in Mathematical Engineering. Given the audience, the treatment is naturally geared toward a mathematically inclined reader. Nevertheless, the required prerequisites are relatively modest, and any student in engineering, mathematics, and statistics should be well-equipped to tackle the contents of this introductory book. The book should also be of interest to students in economics, as well as junior practitioners with a suitable quantitative background.

We begin with quite elementary concepts, and material is introduced progressively, always paying due attention to the practical side of things. Mathematical modeling is an art of selective simplification, which must be supported by intuition building, as well as by a healthy dose of skepticism. This is the aim of remarks, counterexamples, and financial horror stories that the book is interspersed with. Occasionally, we also touch upon current research topics.

**Book structure**

The book is organized into five parts.

1. **Part One, Overview**, consists of two chapters. Chapter 1 aims at getting unfamiliar readers acquainted with the role and structure of financial markets, the main classes of traded assets (equity, fixed income, and derivatives), and the main types of market participants, both in terms of institutions (e.g., investment banks and pension funds) and roles (e.g., speculators, hedgers, and arbitrageurs). We try to give a practical flavor that is essential to students of quantitative disciplines, setting the stage for the application of quantitative models. Chapter 2 overviews the basic problems in finance, like asset allocation, pricing, and risk management, which may be tackled by quantitative models. We also introduce the fundamental concepts related to arbitrage theory, including market completeness and risk-neutral measures, in a simple static and discrete setting.

2. **Part Two, Fixed-income assets**, consists of four chapters and introduces the simplest assets depending on interest rates, starting with plain bonds. The fundamental concepts of interest rate modeling, including the term

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1In case of need, the mathematical prerequisites are covered in my other book: Quantitative Methods: An Introduction for Business Management. Wiley, 2011.
structure and forward rates, as well as bond pricing, are covered in Chapter 3. The simplest interest rate derivatives (forward rate agreements and vanilla swaps) are covered in Chapter 4, whereas Chapter 5 aims at providing the reader with a flavor of real-life markets, where details like day count and quoting conventions are relevant. Chapter 6 concludes this part by showing how quantitative models may be used to manage interest rate risk. In this part, we do not consider interest rate options, which require a stronger mathematical background and are discussed later.

3. Part Three, **Equity portfolios**, consists of four chapters, where we discuss equity markets and portfolios of stock shares. Actually, this is not the largest financial market, but it is arguably the kind of market that the layman is more familiar with. Chapter 7 is a bit more theoretical and lays down the foundations of static decision-making under uncertainty. By static, we mean that we make one decision and then we wait for its consequences, finger crossed. Multistage decision models are discussed later. In this chapter, we also introduce the basics of risk aversion and risk measurement. Chapter 8 is quite classical and covers traditional mean–variance portfolio optimization. The impact of statistical estimation issues on portfolio management motivates the introduction of factor models, which are the subject of Chapter 9. Finally, in Chapter 10, we discuss equilibrium models in their simplest forms, the capital asset pricing model (CAPM), which is related to a single-index factor model, and arbitrage pricing theory (APT), which is related to a multifactor model. We do not discuss further developments in equilibrium models, but we hint at some criticism based on behavioral finance.

4. Part Four, **Derivatives**, includes four chapters. We discuss dynamic uncertainty models in Chapter 11, which is more challenging than previous chapters, as we have to introduce the necessary foundations of option pricing models, namely, stochastic differential equations and stochastic integrals. Chapter 12 describes simple forward and futures contracts, extending concepts that were introduced in Chapter 4, when dealing with forward and futures interest rates. Chapter 13 covers option pricing in the case of complete markets, including the celebrated and controversial Black–Scholes–Merton formula, whereas Chapter 14 extends the basic concepts to the more realistic setting of incomplete markets.

5. Part Five, **Advanced optimization models**, is probably the less standard part of this book, when compared to typical textbooks on financial markets. We deal with optimization model building, in Chapter 15, and optimization model solving, in Chapter 16. Actually, it is difficult to draw a sharp line between model building and model solving, but it is a fact of life that advanced software is available for solving quite sophisticated models, and the average user does not need a very deep knowledge of the involved algorithms, whereas she must be able to build a model. This is the motivation for separating the two chapters.
 Needless to say, the choice of which topics should be included or omitted is debatable and based on authors’ personal bias, not to mention the need to keep a book size within a sensible limit. With respect to introductory textbooks on financial markets, there is a deeper treatment of derivative models. On the other hand, more challenging financial engineering textbooks do not cover, e.g., equilibrium models and portfolio optimization. We aim at an intermediate treatment, whose main limitations include the following:

- We only hint at criticism put forward by behavioral finance and do not cover market microstructure and algorithmic trading strategies.
- From a mathematical viewpoint, we pursue an intuitive treatment of financial engineering models, as well as a simplified coverage of the related tools of stochastic calculus. We do not rely on rigorous arguments involving self-financing strategies, martingale representation theorems, or change of probability measures.
- From a financial viewpoint, by far, the most significant omission concerns credit risk and credit derivatives. Counterparty and liquidity risk play a prominent role in post-Lehman Brothers financial markets and, as a consequence of the credit crunch started in 2007, new concepts like CVA, DVA, and FVA have been introduced. This is still a field in flux, and the matter is arguably not quite assessed yet.
- Another major omission is econometric time series models.

Adequate references on these topics are provided for the benefit of the interested readers.

My choices are also influenced by the kind of students this book is mainly aimed at. The coverage of optimization models and methods is deeper than usual, and I try to open readers’ critical eye by carefully crafted examples and counterexamples. I try to strike a satisfactory balance between the need to illustrate mathematics in action and the need to understand the real-life context, without which quantitative methods boil down to a solution in search of a problem (or a hammer looking for nails, if you prefer). I also do not disdain just a bit of repetition and redundancy, when it may be convenient to readers who wish to jump from chapter to chapter. More advanced sections, which may be safely skipped by readers, are referred to as supplements and their number is marked by an initial “S.”

In my Financial Engineering course, I also give some more information on numerical methods. The interested reader might refer to my other books:

Acknowledgements

In the past years, I have adopted the following textbooks (or earlier editions) in my courses. I have learned a lot from them, and they have definitely influenced the writing of this book:


Other specific acknowledgements are given in the text. I apologize in advance for any unintentional omission.

Additional material

Some end-of-chapter problems are included and fully worked solutions will be posted on a web page. My current URL is

- http://staff.polito.it/paolo.brandimarte/

A hopefully short list of errata will be posted there as well. One of the many corollaries of Murphy’s law states that my URL is going to change shortly after publication of the book. An up-to-date link will be maintained on the Wiley web page:

- http://www.wiley.com/

For comments, suggestions, and criticisms, all of which are quite welcome, my e-mail address is

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*Paolo Brandimarte*

*Turin, September 2017*