Recenti risultati sulle applicazioni delle teorie cinetiche alla dinamica delle folle

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Workshop: Kinetic theory methods toward applications
Torino, December 16th, 2013

Complexity features of a crowd

- Ability to express a strategy: Living entities are capable to develop specific *strategies* (trend toward the exit, avoiding clusters, avoiding walls and obstacles, perception of signals, etc).
- Heterogeneity: The ability described in the first item is *heterogeneously distributed.* Heterogeneity includes, in addition to different walking abilities, also the possible presence of a hierarchy (namely a leader).
- Interactions: *Interactions are nonlinearly additive* and are *nonlocal in space*, since pedestrians perceive stimuli at a distance which depends on the geometry of the system where they move, as well as on the general environmental and psychological conditions.

From individuality to collectivity


The mean distance between pedestrians may be small or large, where pedestrians either fill the whole square or walk in streets without overcrowding phenomena. In some cases these limit situations can occur within the same area.

Hallmarks of the kinetic theory of active particles

- The overall system is subdivided into *functional subsystems* constituted by entities, called *active particles*, whose individual state is called *activity*;
- The state of each functional subsystem is defined by a suitable, time dependent, *distribution function over the microscopic state*;
- Interactions are modeled by tools of games theory, more precisely stochastic games, where the state of the interacting particles and the output of the interactions are known in probability;
- Interactions are delocalized and nonlinearly additive;
- The evolution of the distribution function is obtained by a balance of particles within elementary volumes of the space of the microscopic states, where the dynamics of inflow and outflow of particles is related to interactions at the microscopic scale.

Toward a kinetic theory of active pedestrians

	Crowd dynamics
Active particles	Pedestrians
Microscopic state	Position Velocity Activity
Functional subsystems	Different abilities Individuals pursuing different targets etc.

Toward a kinetic theory of active pedestrians

The corresponding representation of the system by means of the kinetic theory needs the assessment of the distribution function over the micro-state:

$$f : [0,T] \times \Sigma \times D_{\mathbf{v}} \times D_u \to \mathbb{R}_{\geq 0}, \tag{1}$$

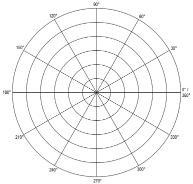
such that $f(t,\mathbf{x},\mathbf{v},u)d\mathbf{v}du$ denotes the infinitesimal number of pedestrians that, at time t, are situated in the position $\mathbf{x}$ and with micro-state in the interval $[\mathbf{v},\mathbf{v}+d\mathbf{v}] \times [u,u+du]$.
Macroscopic quantities are obtained by weighted moments of f. In particular, the local density is given by:

$$\rho(t,\mathbf{x}) = \int_{D_u}\int_{D_{\mathbf{v}}} f(t,\mathbf{x},\mathbf{v})d\mathbf{v}du. \tag{2}$$

Description of the micro-state

- Since the dynamics takes place in a two dimensional domain, the velocity variable is described by a vector $\mathbf{v} \in \mathbb{R}^2$.
- Polar coordinates for the velocity variable

$$\mathbf{v} = \begin{cases} \mathit{speed} & \rightarrow \quad v \in [0,1] \\ \mathit{direction} & \rightarrow \quad \theta \in [0, 2\pi) \end{cases}$$



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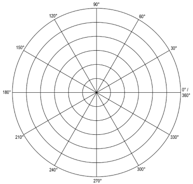

- The directions can be discretized

$$I_\theta = \{\theta_1 = 0, \dots, \theta_i, \dots, \theta_n = \frac{n-1}{n} 2\pi\}$$

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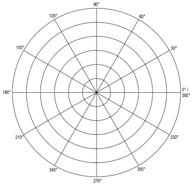

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$$I_\theta = \{\theta_1 = 0, \ldots, \theta_i, \ldots, \theta_n = \frac{n-1}{n} 2\pi\} \quad \rightarrow f_i(t,\mathbf{x},v,u) = f(t,\mathbf{x},v,\theta_i,u)$$

What about the speed?

Different approaches may be considered:
- The velocity modulus can be discretized

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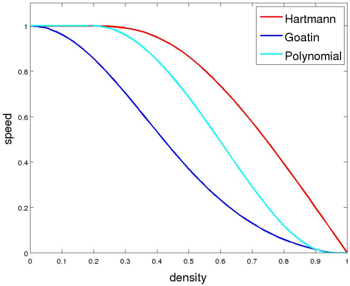
- Or, alternatively, it can be regarded as a variable that depends on macroscopic issues, motivated by the fact that the speed of pedestrians strongly depends on the level of congestion around them.

$$v = v(\rho)$$



$$v(\rho) = v_M \left(1 - \exp\left(-\gamma\left(\frac{1}{\rho} - \frac{1}{\rho_M}\right)\right)\right)^{1}$$

$$v(\rho) = v_M \exp\left(-\gamma\left(\frac{\rho}{\rho_M}\right)^2\right)^{2}$$

[1] Hartmann, Von Sivers (2013)

[2] Buchmueller, Weidmann (2006) - Goatin *et al.* (2013)

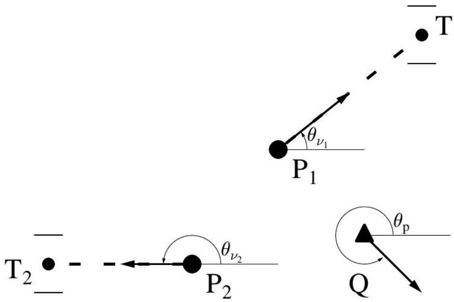
- **Perceived density**: The main idea is that the particle perceives a local density different from the real one, namely it feels a higher density if the density is increasing along its direction or the two adjacent directions, while it perceives a lower density in the other case. In details, let us consider a pedestrian at $\mathbf{x}$, moving with direction θ_i. We define the perceived density [3] as

$$\rho_i^P[\rho](t,\mathbf{x}) = \rho(t,\mathbf{x}) + \frac{\partial_i\rho(t,\mathbf{x})}{\sqrt{1+\partial_i\rho(t,\mathbf{x})^2}}\left[(1-\rho(t,\mathbf{x}))H(\partial_i\rho(t,\mathbf{x})) + \rho(t,\mathbf{x})H(-\partial_i\rho(t,\mathbf{x}))\right] \tag{3}$$

where ∂_i denotes the directional derivative along the direction θ_i, and H is the Heaviside function, i.e. $H(x) = 1$ if $x \geq 0$, and $H(x) = 0$ if $x < 0$.

[3] Bellomo, Bellouquid, Nieto, Soler (2013)

Crowds in unbounded domains



Particles in P_1 and P_2 move, respectively, toward the directions T_1 and T_2 identified by the directions θ_{ν_1} and θ_{ν_2} with respect to the horizontal axis. [4]

[4] Bellomo, Bellouquid, Knopoff (2013)

Description of the micro-state

$$f(t,\mathbf{x},\mathbf{v},u) = \sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}(t,\mathbf{x},u)\,\delta(\theta-\theta_i)\otimes\delta(v-v_j)$$

Some specific cases can be considered. For instance the case of two different groups, labeled with the superscript $\sigma = 1,2$, which move towards two different targets.

$$f^{\sigma}(t,\mathbf{x},\mathbf{v},u) = \sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}^{\sigma}(t,\mathbf{x})\,\delta(\theta-\theta_i)\otimes\delta(v-v_j)\otimes\delta(u-u_0)$$

Mathematical structures

Variation rate of the number of active particles	=	Inlet flux rate caused by conservative interactions	−	Outlet flux rate caused by conservative interactions

$$
\begin{aligned}
(\partial_t \quad &+ \quad \mathbf{v}_{ij} \cdot \partial_{\mathbf{x}}) f_{ij}^{\sigma}(t, \mathbf{x}) = J[\mathbf{f}](t, \mathbf{x}) \\
&= \sum_{h,p=1}^{n} \sum_{k,q=1}^{m} \int_{\Lambda} \eta[\rho(t, \mathbf{x}^*)] \mathcal{A}_{hk,pq}^{\sigma}(ij)[\rho(t, \mathbf{x}^*)] f_{hk}^{\sigma}(t, \mathbf{x}) f_{pq}^{\sigma}(t, \mathbf{x}^*) \, d\mathbf{x}^* \\
&- \quad f_{ij}^{\sigma}(t, \mathbf{x}) \sum_{p=1}^{n} \sum_{q=1}^{m} \int_{\Lambda} \eta[\rho(t, \mathbf{x}^*)] f_{pq}^{\sigma}(t, \mathbf{x}^*) \, d\mathbf{x}^*, \qquad (4)
\end{aligned}
$$

where $\mathbf{f} = \{f_{ij}\}$.

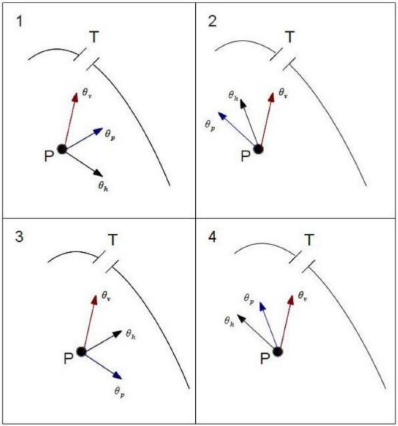
Interaction terms

• **Interaction rate:** It is assume that it increases with increasing local density in the free and congested regimes. For higher densities, when pedestrians are obliged to stop, one may assume that it keeps a constant value, or may decay for lack of interest.

• **Transition probability density:** We assume that particles are subject to two different influences, namely the *trend to the exit point* and the *influence of the stream* induced by the other pedestrians. A simplified interpretation of the phenomenological behavior is obtained by assuming the factorization of the two probability densities modeling the modifications of the velocity direction and modulus:

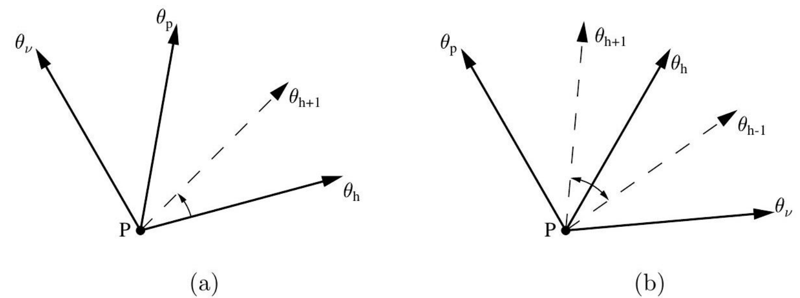
$$\mathcal{A}^{\sigma}_{hk,pq}(ij) = \mathcal{B}^{\sigma}_{hp}(i)\big(\theta_h \to \theta_i|\rho(t,\mathbf{x})\big) \times \mathcal{C}^{\sigma}_{kq}(j)\big(v_k \to v_j|\rho(t,\mathbf{x})\big).$$

Interactions in the table of games



Particle in P moves to a direction θ_h (black arrow) and interacts with a field particle moving to θ_p (blue arrow), the direction to the target is θ_v (red arrow).

Interactions in the table of games



A particle can change its velocity direction, in probability, only to an **adjacent state**. (a) A candidate particle with direction θ_h interacts with an upper stream with direction θ_p and target directions θ_ν and decides to change its direction to θ_{h+1}. (b) A candidate particle interacts with an upper stream and lower target directions, and decides to change its direction either to θ_{h+1} or θ_{h-1}.

Transition probability density (example)

– *Interaction with an upper stream and target directions, namely* $\theta_p > \theta_h, \quad \theta_\nu > \theta_h$*:*

$$\mathcal{B}_{hp}(i) \begin{cases} \alpha u_0(1-\rho) + \alpha u_0 \rho & \text{if} \quad i = h+1, \\ 1 - \alpha u_0(1-\rho) - \alpha u_0 \rho & \text{if} \quad i = h, \\ 0 & \text{if} \quad i = h-1, \end{cases}$$

$$\alpha = \text{quality of the environment}$$

Mild form of the initial value problem

The initial value problem consists in solving Eqs. (4) with initial conditions given by

$$f_{ij}^{\sigma}(0,\mathbf{x}) = \phi_{ij}^{\sigma}(\mathbf{x}).$$

Let us introduce the mild form obtained by integrating along the characteristics:

$$\begin{aligned}\widehat{f_{ij}^{\sigma}}(t,\mathbf{x}) &= \phi_{ij}^{\sigma}(\mathbf{x}) + \int_0^t \left(\widehat{\Gamma_{ij}^{\sigma}}[\mathbf{f},\mathbf{f}](s,\mathbf{x}) - \widehat{f_{ij}^{\sigma}}(s,\,\mathbf{x}) \widehat{Ł[\mathbf{f}]}(s,\,\mathbf{x}) \right) ds,\\ & i \in \{1,\ldots,n\}, \quad j \in \{1,\ldots,m\}, \quad \sigma \in \{1,2\},\end{aligned}$$

where the following notation has been used for any given vector $f(t,\mathbf{x})$: $\widehat{f_{ij}^{\sigma}}(t,\mathbf{x}) = f_{ij}^{\sigma}(t, x + v_j \cos(\theta_i)t, y + v_j \sin(\theta_i)t)$.

Existence theory

H.1. For all positive R, there exists a constant $C_\eta > 0$ so that $0 < \eta(\rho) \le C_\eta\,, \quad \text{whenever} \quad 0 \le \rho \le R.$

H.2. Both the encounter rate $\eta[\rho]$ and the transition probability $\mathcal{A}^{\sigma}_{hk,pq}(ij)[\rho]$ are Lipschitz continuous functions of the macroscopic density ρ, i.e., that there exist constants $L_\eta, L_{\mathcal{A}}$ is such that

$$|\,\eta[\rho_1] - \eta[\rho_2]\,| \le L_\eta\,|\,\rho_1 - \rho_2\,|,$$

$$|\,\mathcal{A}^{\sigma}_{hk,pq}(ij)[\rho_1] - \mathcal{A}^{\sigma}_{hk,pq}(ij)[\rho_2]\,| \le L_{\mathcal{A}}\,|\,\rho_1 - \rho_2\,|,$$

whenever $0 \le \rho_1 \le R$, $0 \le \rho_2 \le R$, and all $i, h, p = 1, .., n$ and $j, k, q = 1, .., m$.

• Let $\phi_{ij}^{\sigma} \in L^{\infty} \cap L^{1}$, $\phi_{ij}^{\sigma} \geq 0$, then there exists ϕ^0 so that, if $\| \phi \|_1 \leq \phi^0$, there exist T, a_0, and R so that a unique non-negative solution to the initial value problem exists and satisfies:

$$f \in X_T, \qquad \sup_{t \in [0,T]} \| f(t) \|_1 \leq a_0 \| \phi \|_1,$$

$$\rho(t, \mathbf{x}) \leq R, \quad \forall t \in [0,T], \quad \mathbf{x} \in \Omega.$$

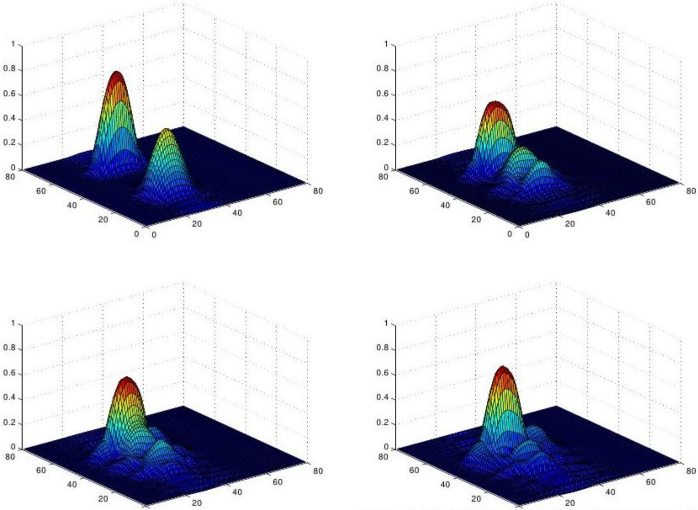
Moreover, if $\sum_{\sigma=1}^{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \| \phi_{ij}^{\sigma} \|_{\infty} \leq 1$, and $\| \phi \|_1$ is small, one has $\rho(t, \mathbf{x}) \leq 1, \quad \forall t \in [0,T], \quad \mathbf{x} \in \Omega$.

• There exist ϕ^r, $(r = 1, \ldots, p-1)$ such that if $\| \phi \|_1 \leq \phi^r$, there exists a_r so that it is possible to find a unique non-negative solution to the initial value problem satisfying for any $r \leq p-1$ the following $f(t) \in X[0,(p-1)T]$,

$$\sup_{t \in [0,T]} \| f(t + (r-1)T) \|_1 \leq a_{r-1} \| \phi \|_1,$$

and $\rho(t + (r-1)T, \mathbf{x}) \leq R, \quad \forall t \in [0,T], \quad \mathbf{x} \in \Omega$. Moreover, $\rho(t + (r-1)T, \mathbf{x}) \leq 1, \quad \forall t \in [0,T], \quad \mathbf{x} \in \Omega$.

Case-study 1



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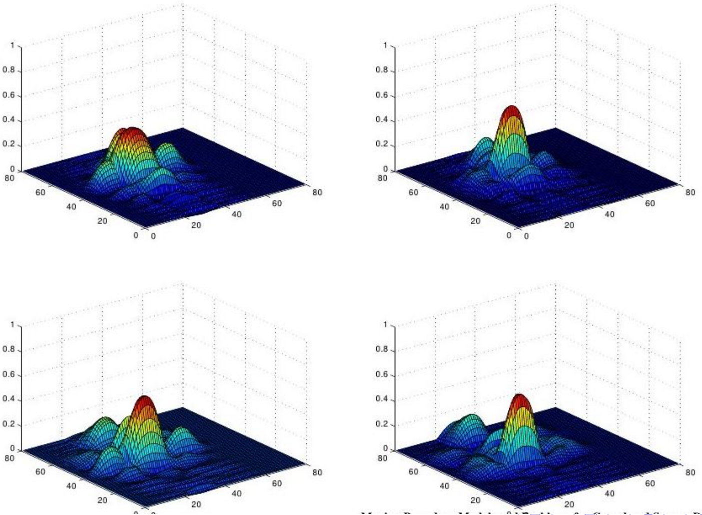


An approach to bounded domains

Goal

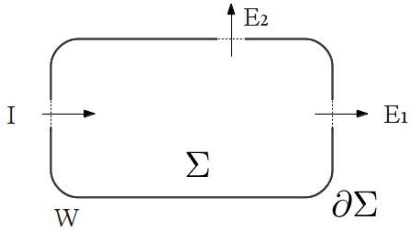
The aim of this work [a] is to model the dynamics of a crowd in a bounded domain taking into account:

- interactions with walls
- flow through entrance/exit doors

[a] Agnelli, Colasuonno, Knopoff (2013)

Description of the system

- Bounded domain $\Sigma \subset \mathbb{R}^2$.
- Boundary $\partial\Sigma$:
  - $I \subset \partial\Sigma$ inlet zone (entrance)
  - $E \subset \partial\Sigma$ outlet zone (exit)
  - $W = \partial\Sigma - (E \cup I)$ wall.



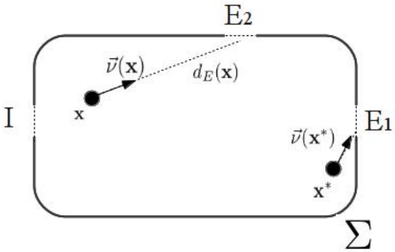
Interaction dynamics

We take into account four different effects:

1. Trend to move toward the exit.
2. Trend to avoid the collision with walls.
3. Tendency to avoid congested areas.
4. Tendency to follow the stream.

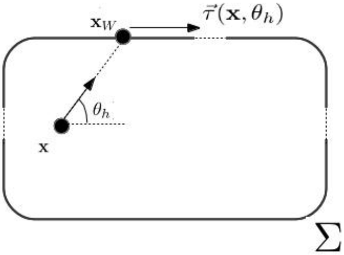
1 Trend to move toward the exit.

- $d_E(\mathbf{x})$ = distance from $\mathbf{x}$ to the exit E.
- $\vec{v}(\mathbf{x})$ = unitary vector pointing from $\mathbf{x}$ to E.



2 Trend to avoid the collision with walls.

- $d_W(\mathbf{x},\theta_h)$ = distance from $\mathbf{x}$ to the wall W in the direction θ_h.
- $\vec{\tau}(\mathbf{x};\theta_h)$ is the unitary tangent vector at the point $\mathbf{x}_W$.



3 Tendency to move towards less congested areas.

- In order to facilitate its movement, a candidate particle in $\mathbf{x}$ moving with direction θ_h may decide to change direction by considering the presence of less congested areas.
- $\vec{\gamma}(\mathbf{x};\rho) =$ direction with the largest decrease of the density.

3 Tendency to follow the stream.

- A candidate h-particle interacts with a field p-particle, moving with direction θ_p, and may decide to follow it.
- $\vec{\sigma}_p = (\cos\theta_p, \sin\theta_p)$ is the stream vector.

Description of the system

Effect	Versor	Geometry	Density
exit	$\vec{\nu}(\mathbf{x})$	$1 - d_E(\mathbf{x})$	$1 - \rho(t, \mathbf{x})$
wall	$\vec{\tau}(\mathbf{x}, \theta_h)$	$1 - d_W(\mathbf{x}, \theta_h)$	$1 - \rho(t, \mathbf{x})$
vacuum	$\vec{\gamma}(\mathbf{x}; \rho)$	$d_E(\mathbf{x})$	$\rho(t, \mathbf{x})$
stream	$\vec{\sigma}_k$	1	$\rho(t, \mathbf{x})$

Mathematical structure

$$(\partial_t + \mathbf{v}_i[\rho_i](t,\mathbf{x}) \cdot \nabla_{\mathbf{x}}) f_i(t,\mathbf{x}) = \varepsilon_1 \mathcal{J}_i^B[\mathbf{f}](t,\mathbf{x}) + (1-\varepsilon_1)\mathcal{J}_i^M[\mathbf{f}](t,\mathbf{x}),$$

where

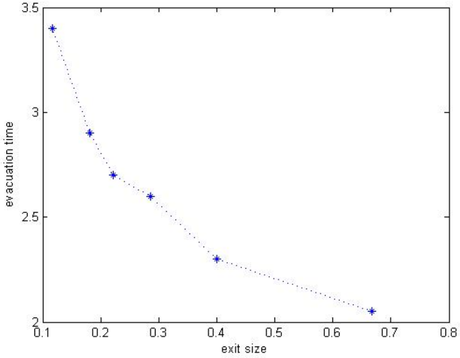
$$\begin{aligned} \mathcal{J}_i^B[\mathbf{f}](t,\mathbf{x}) &= \eta[\rho(t,\mathbf{x})] \sum_{h,k=1}^{n} \mathcal{A}_{hk}(i)[\mathbf{f}] f_h(t,\mathbf{x}) f_k(t,\mathbf{x}) \\ &- f_i(t,\mathbf{x})\eta[\rho(t,\mathbf{x})]\rho(t,\mathbf{x}) \end{aligned}$$

$$\begin{aligned} \mathcal{J}_i^M[\mathbf{f}](t,\mathbf{x}) &= \mu[\rho(t,\mathbf{x})] \sum_{h=1}^{n} \mathcal{B}_h(i)[\mathbf{f}] f_h(t,\mathbf{x}) \\ &- f_i(t,\mathbf{x})\mu[\rho(t,\mathbf{x})] \end{aligned}$$

Numerical results

Numerical results

Exit time vs. door size



• **J.P. Agnelli, F. Colasuonno, and D. Knopoff**, A kinetic theory approach to the modeling of crowd dynamics and evacuation in bounded domains, under preparation.

• **N. Bellomo and C. Dogbè**, On the modelling of traffic and crowds - a survey of models, speculations, and perspectives, *SIAM Review*, 53(3) (2011), 409–463.

• **N. Bellomo and J. Soler**, On the mathematical theory of the dynamics of swarms viewed as complex systems, *Math. Models Methods Appl. Sci.*, 22 Supp. 1 (2012), 1140006.

• **N. Bellomo, A. Bellouquid, and D. Knopoff**, From the micro-scale to collective crowd dynamics, *SIAM Multiscale Model. Simul.*, 11(3) (2013), 943–963.

Questions?