Multiscale Methods for Crowd Dynamics

Individuality vs. Collectivity

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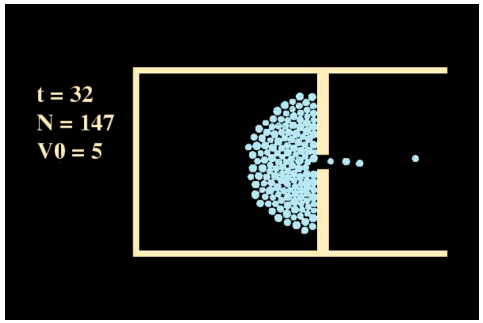
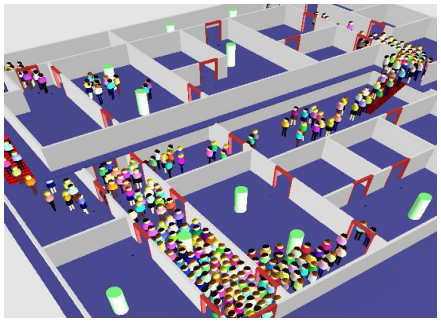
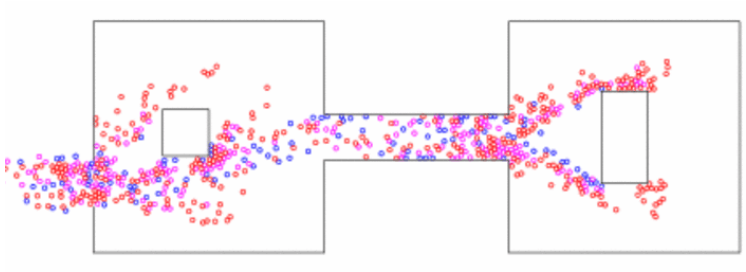
Microscopic (Particle-Based) Models

"Social force" model (Helbing *et al.*, 1995)

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \dfrac{v_{0,i} - v_i}{\tau} - \displaystyle\sum_{j=1}^{N} \nabla U_{ij}(x_j - x_i) + \dots \end{cases}$$







"Contact handling" model (Maury and Venel, 2007)

$$\dot{x}_i = \mathcal{P}_{C(X)}(V_{\text{des}}(x_i))$$

- $V_{\text{des}} : \mathbb{R}^d \to \mathbb{R}^d$ pedestrian desired velocity
- $\mathcal{P}_{C(X)}$ projection operator on the space of admissible velocities

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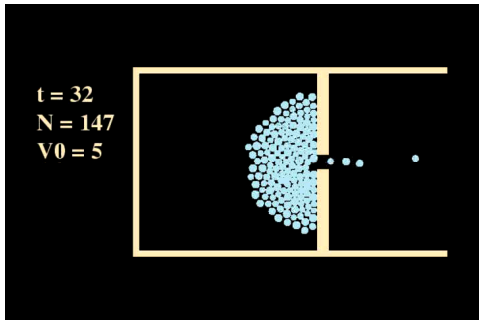

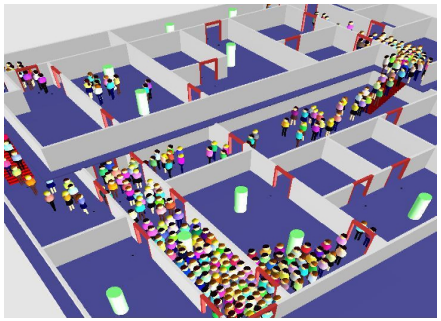

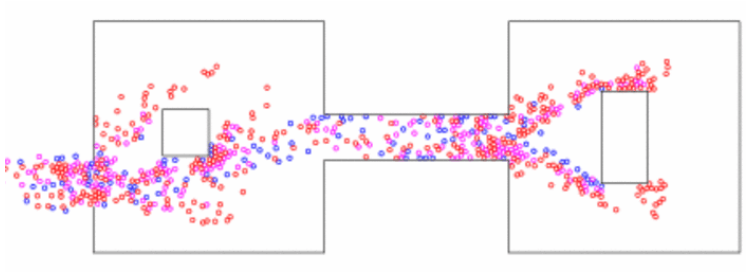

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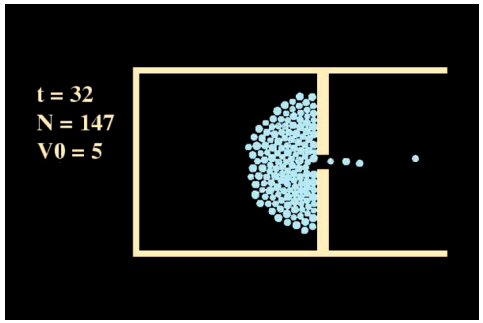

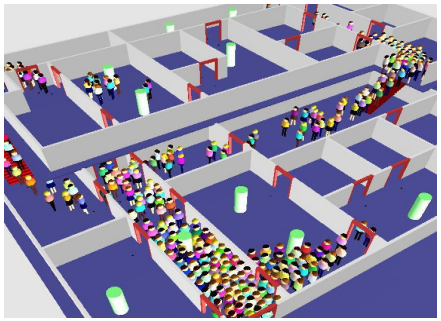

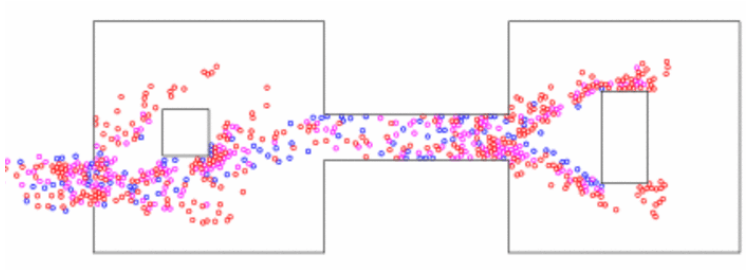

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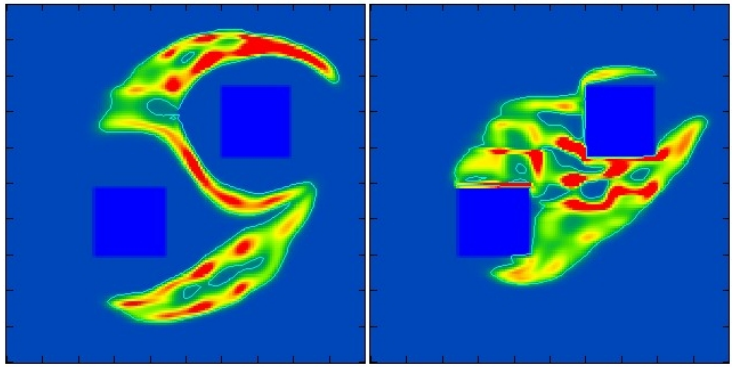
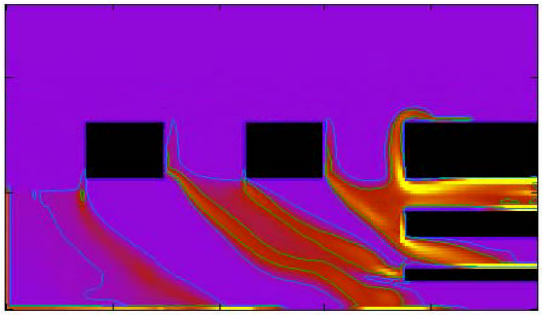
Macroscopic (Density-Based) Models

First order model (Hughes 2002, Colombo *et al.* 2005, Bruno-Venuti 2007, Coscia-Canavesio 2008, ...)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V(\rho) \boldsymbol{\nu}) = 0$$

- $V(\rho)$ speed-density relationship
- $\boldsymbol{\nu}$ preferred direction of movement





Second order model (Bellomo-Dogbé 2008, Coscia-Canavesio 2008, Twarogowska *et al.* 2013)

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \\ \dfrac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho v \otimes v) = \dfrac{\rho V(\rho) \boldsymbol{\nu} - \rho v}{\tau} - \nabla P(\rho) \end{cases}$$

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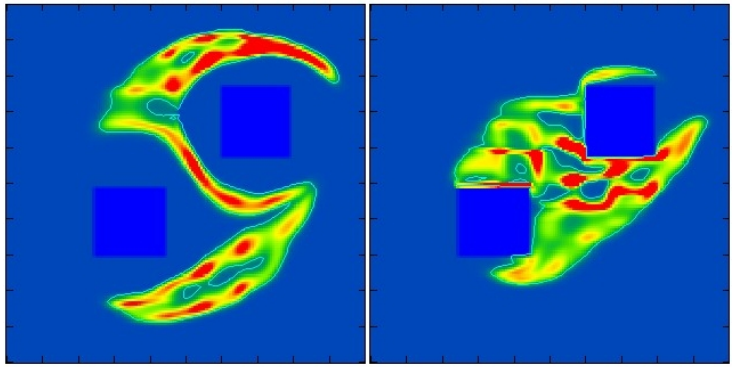

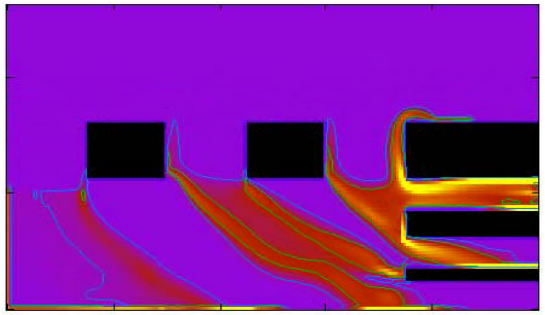

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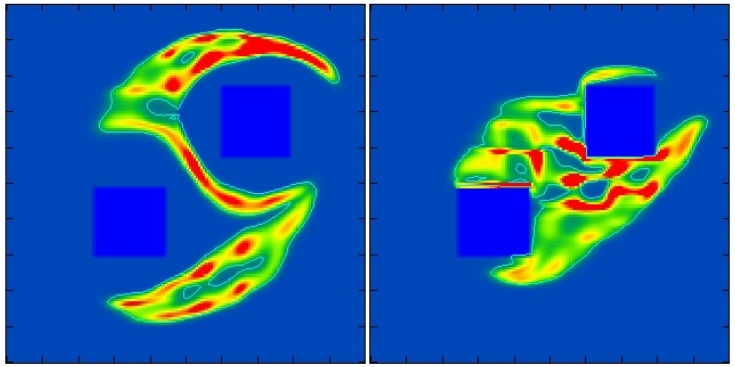

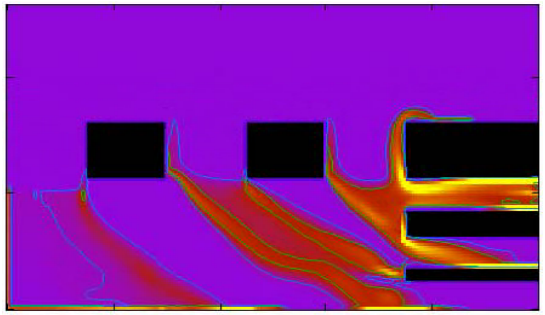

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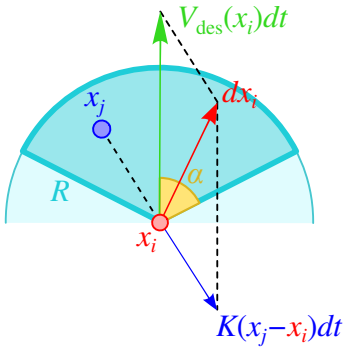
Multiscale Descriptive Approach by Flow Maps and Measures

- Get inspiration from a simple first order particle model:

$$\dot{x}_i = V_{\text{des}}(x_i) + \sum_{x_j \in S_{R,\alpha}(x_i)} K(x_j - x_i)$$



- Label pedestrians by means of their initial position:

$$X_t : \mathbb{R}^d \to \mathbb{R}^d \quad \text{(flow map)}$$

$$x = X_t(\xi) : \text{ position at time } t > 0 \text{ of the walker who initially was in } \xi \in \mathbb{R}^d$$

- Fix a walker ξ and rewrite the particle model using the flow map:

$$\dot{X}_t(\xi) = V_{\text{des}}(X_t(\xi)) + \int_{X_t^{-1}(S_{R,\alpha}(X_t(\xi)))} K(X_t(\eta) - X_t(\xi))\, d\mu_0(\eta)$$

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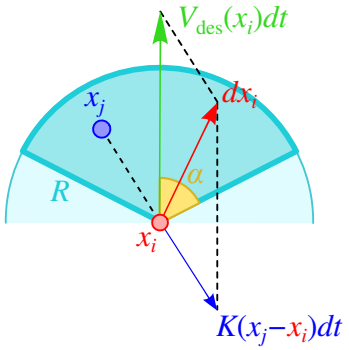

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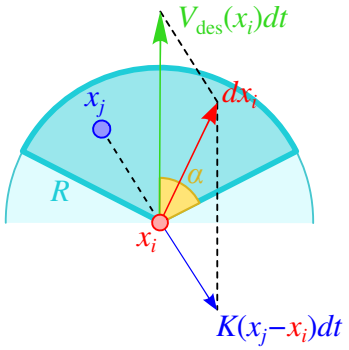

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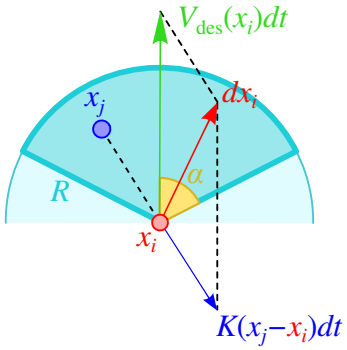

- Transport μ_0 by means of X_t, i.e., $\boxed{\mu_t := X_t \# \mu_0}$, to discover:

$$\begin{cases} \dfrac{\partial \mu}{\partial t} + \nabla \cdot (\mu v[\mu]) = 0 \\ v[\mu_t](x) = V_{\text{des}}(x) + \displaystyle\int_{S_{R,\alpha}(x)} K(y - x)\, d\mu_t(y) \end{cases} \qquad x \in \mathbb{R}^d,\ t > 0$$

- Description compatible with both a discrete and a continuous view of the crowd:

discrete: $\boxed{\mu_0 = \sum_{i=1}^{N} \delta_{\xi_i}}$, continuous: $\boxed{\mu_0 = \rho_0 \mathcal{L}^d}$

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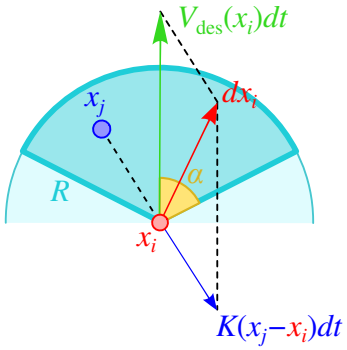

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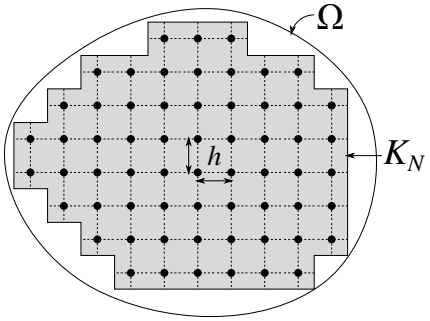
Discrete and Continuous: So Close…

- Comparison between discrete and continuous models in terms of statistical distributions of pedestrians:

$$\mu_0^{\text{discr}} = \boxed{\frac{1}{N}} \sum_{i=1}^{N} \delta_{\xi_i}, \qquad \mu_0^{\text{cont}} = \boxed{\frac{1}{\mathcal{L}^d(\Omega)}} \mathcal{L}^d$$

- 

$\rightsquigarrow \; W_1(\mu_0^{\text{discr}}, \mu_0^{\text{cont}}) \leq \frac{\sqrt{d}}{2} h + \frac{\text{diam}(\Omega)}{\mathcal{L}^d(\Omega)} \mathcal{L}^d(\Omega \setminus K_N) \xrightarrow{N \to \infty} 0$

- Continuous dependence estimate (for smooth V_{des} and K):

$$W_1(\mu_t^{\text{discr}}, \mu_t^{\text{cont}}) \leq \mathcal{C}\, W_1(\mu_0^{\text{discr}}, \mu_0^{\text{cont}}) \quad \forall\, t \in (0, T]$$

hence $\boxed{\mu_t^{\text{discr}} \equiv \mu_t^{\text{cont}}}$ for all t in the limit $N \to \infty$.

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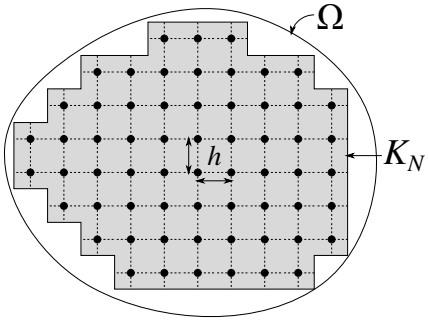
- 

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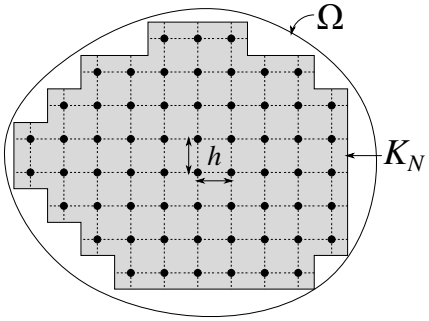
- 

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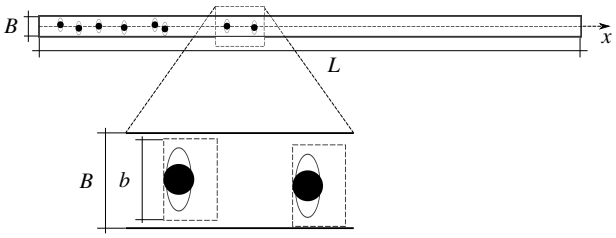
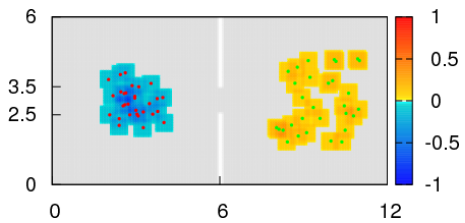
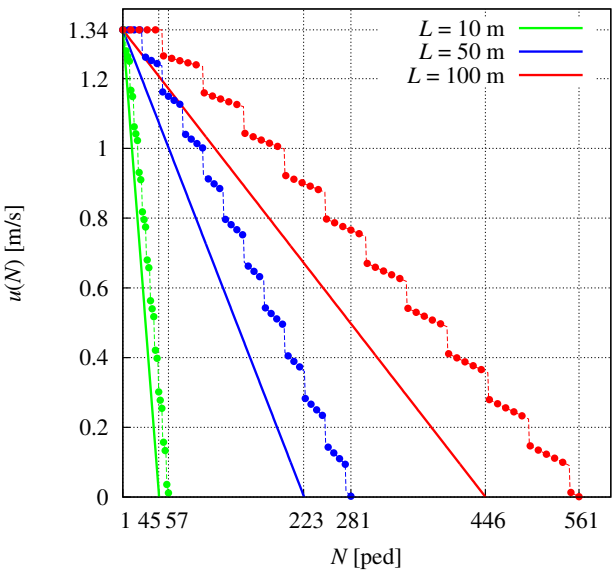
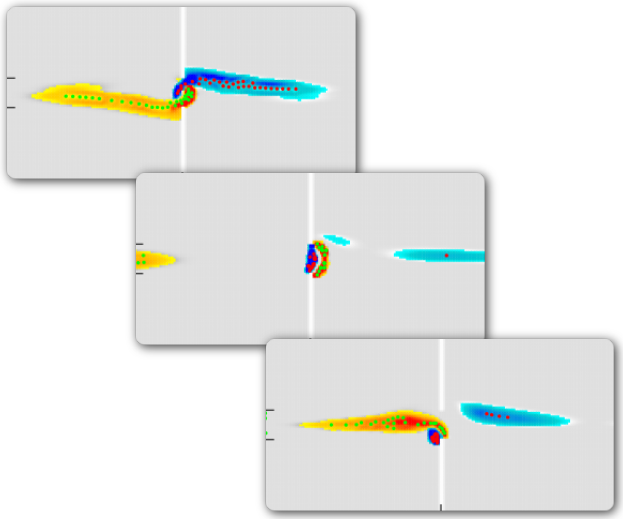
Discrete and Continuous: . . . So Far

- Does the limit $N \to \infty$ really make sense for crowds?
  - Often it does not: pedestrians in a crowd are not as many as 10^{23} gas molecules.
  - For finite N physical mass distributions matter.









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