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Structural Interpretation of Transmission Zeros for Matrix Second-order Systems*

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Key Words—Transmission zeros; multivariable systems; sensors; actuators; mechanical systems.

Abstract—A novel result is presented that relates the transmission zeros (TZ) of a multivariate system described in matrix second-order form to the constrained natural frequencies (poles) of the system itself. The constrained system defines autonomous and energetically isolated subsystems, whose eigenfrequencies are the TZ of the system. The result is valid for systems with colocated input–output devices, and can be extended to the non-colocated case under some more restrictive assumptions. Into the considered class of systems fall lumped parameters mechanical structures, whose finite element models (FEM) are particularly suited for the use of the proposed method. © 1997 Elsevier Science Ltd.

1. *Introduction*

The motivating questions of this paper are the following. What is the physical significance of the transmission zeros (TZ) of a multi-input multi-output (MIMO) structure? How are TZ related to the classical concept of structural 'antiresonance', well known for single-input single-output (SISO) systems? This paper shows that for systems described in second-order matrix polynomial form, TZ are the poles of the system itself, when constraints are put at all the sensor and actuator colocated locations. Some important features of SISO colocated structural systems, as the pole–zero interlacing property (Martin and Bryson, 1980), are also recovered.

A number of researchers have studied various aspects of the TZ problem, both in theory, (e.g. Rosenbrock, 1970; Desoer and Shulman, 1974; Kouvaritakis and MacFarlane, 1976) and in applications (see e.g. Maghami and Joshi, 1993, and references therein). However, very little has been said about the physical significance of the zeros and their relation to the poles of a system. A first attempt in this direction can be found in Miu (1991) and Miu and Yang (1994), where it is shown that for a SISO mechanical systems, transfer-function zeros could be found as resonances of the structure constrained at the sensor and actuator locations. This introduced the important idea that the zero frequencies are determined by only a small part of the system, which behaves autonomously, and where the energy of the system is 'trapped'. The result of Miu (1991) was limited to SISO systems with no damping, and is extended here to the general MIMO case. The identification of the zero-providing subsystems is presented in Sections 4.1 and 4.2 for colocated and non-colocated systems respectively. Computational issues are also discussed in Section 4.1.

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2. *Preliminaries and definitions*

The first-order model of a linear, finite-order, time-invariant and proper MIMO system is given by the well-known equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \tag{1}$$

where $\mathbf{A}$ is the $N \times N$ real system matrix, $\mathbf{B}$ is an $N \times l$ real matrix, $\mathbf{C}$ is an $m \times N$ real matrix, $\mathbf{x}$ is the $N \times 1$ state vector, $\mathbf{u}$ is the $l \times 1$ input vector and $\mathbf{y}$ is the $m \times 1$ output vector. The transfer-function matrix of the system is given by $\mathbf{F}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, and it will be assumed for the rest of the paper that the sytstem is square (with an equal number m of sensors and actuators), and completely controllable and observable (CCCO). These assumptions avoid ambiguity in the definition of TZ, since the set of invariant zeros (MacFarlane and Karcanias, 1976), computed from the Rosenbrock system matrix, and the set of transmission zeros, computed from the invariant factors of the transfer-function matrix via the Smith–McMillan decomposition, are the same. Moreover, it has been shown (Chen 1968) that, under these hypotheses, the pole polynomial of $\mathbf{F}$ (computed as the least common denominator of all non-zero minors of all orders of $\mathbf{F}(s)$) coincides with $\det(s\mathbf{I} - \mathbf{A})$, and will be denoted simply by $p(s)$ without distinction. Also $\mathbf{F}(s)$ will be assumed to have full normal rank, thus ensuring that its determinant is non-zero, except for a finite number of values of s (Desoer and Shulman, 1974).

The following well-known relations between a matrix $\mathbf{D} \in \mathbb{C}^{n,n}$ and its adjugate (or classical adjoint) $\mathbf{N} = \operatorname{adj}\mathbf{D} \in \mathbb{C}^{n,n}$ are recalled. Denoting by $|\mathbf{D}|$ the determinant of $\mathbf{D}$, then $\mathbf{D}^{-1} = \operatorname{adj}\mathbf{D}/|\mathbf{D}| = \mathbf{N}/|\mathbf{D}|$, and the following important relation holds between the minors of $\mathbf{D}$ and $\mathbf{N}$. Consider two ordered index sets $\alpha = \{i_1, i_2, \ldots, i_m\}$ and $\beta = \{j_1, j_2, \ldots, j_m\}$, $m \leq n$, and the complementar sets $\bar{\alpha} = \{i_{m+1}, \ldots, i_n\}$ and $\bar{\beta} = \{j_{m+1}, \ldots, j_n\}$. We denote by $\mathbf{D}_{\alpha,\beta}$ the matrix obtained by crossing the α rows and the β columns of $\mathbf{D}$, i.e. the submatrix that lies in the rows of $\mathbf{D}$ indexed by α and the columns indexed by β. The complement of $\mathbf{D}_{\alpha,\beta}$ will be $\mathbf{D}_{\bar{\alpha},\bar{\beta}}$, which can be obtained from $\mathbf{D}$ by deleting the α rows and the β columns. The determinant of each of these submatrices is called a minor of the $\mathbf{D}$ matrix. Let $\xi = \sum_{k=1}^{m} (i_k + j_k)$; then (Horn and Johnson, 1985)

$$|\mathbf{N}_{\alpha,\beta}| = (-1)^{\xi}\, |\mathbf{D}|^{m-1}\, |\mathbf{D}_{\bar{\beta},\bar{\alpha}}|\,. \tag{2}$$

Definition of transmission zeros. The zero frequencies of the linear multivariate system (1) can be characterized (Kailath, 1980) as those values s_0 of the complex frequency s for which there exist a vector $\mathbf{u}_0$ and an initial state $\mathbf{x}_0$ such that for the input $\mathbf{u}(t) = \mathbf{u}_0 e^{s_0 t}$, $t \geq 0$, the output response is identically zero: $\mathbf{y}(t) \equiv \mathbf{0}$ $\forall t > 0$. The above condition is consistent with the following definition (Rosenbrock, 1970). Consider the Rosenbrock system matrix for the system (1):

$$\mathbf{P}(s) = \begin{bmatrix} s\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & 0 \end{bmatrix}; \quad (3)$$

the *zero polynomial* of the system is defined as the determinant of the system matrix: $z(s) = |\mathbf{P}(s)|$, and the TZ of the system are the roots of the zero polynomial. Using Schur complements, the above equation can be rewritten as $z(s) = |s\mathbf{I} - \mathbf{A}|\,|\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}| = p(s)\,|\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}|$, which, applying the adjoint formula, becomes

$$z(s) = \frac{1}{p^{m-1}(s)} |\mathbf{C}\,\mathrm{adj}\,(s\mathbf{I} - \mathbf{A})\mathbf{B}|. \quad (4)$$

3. *Input–output description of MIMO second-order systems*

A second order linear multivariate system is here described by the equations

$$\mathbf{U}\ddot{\mathbf{q}} + \mathbf{V}\dot{\mathbf{q}} + \mathbf{Z}\mathbf{q} = \mathbf{E}_\alpha \mathbf{H}\mathbf{u}, \quad (5)$$

$$\mathbf{y} = \mathbf{C}_\mathrm{p}\mathbf{E}'_\beta \mathbf{q} + \mathbf{C}_\mathrm{v}\mathbf{E}'_\beta \dot{\mathbf{q}}, \quad (6)$$

where $\mathbf{q} \in \mathbb{R}^n$ is the vector of generalized coordinates, $\mathbf{u} \in \mathbb{R}^m$ is the input forcing term vector, $\mathbf{y} \in \mathbb{R}^m$ is the output vector, $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{Z}$ are $n \times n$ real matrices, $\mathbf{C}_\mathrm{p}$, $\mathbf{C}_\mathrm{v} \in \mathbb{R}^{m,m}$ are non-singular matrices of output influence coefficients, $\mathbf{H} \in \mathbb{R}^{m,m}$ is the non-singular matrix of input influence coefficients, and $E_\alpha \in \mathbb{R}^{n,m}$ and $\mathbf{E}_\beta \in \mathbb{R}^{n,m}$ are matrices that reflect the location on the system of the input and output devices respectively. A very important class of systems that can be described by (5) is that of lumped-parameters flexible mechanical structures, for which $\mathbf{U} > 0$ (symmetric positive-definite) is the mass matrix, and $\mathbf{V} \geq 0$ and $\mathbf{Z} \geq 0$ are the damping and stiffness matrices respectively (Meirovitch, 1990).

If $m \leq n$ locations for the input and output devices are selected, the relative indices can be grouped into two ordered sets: the input set $\alpha = \{i_1, i_2, \ldots, i_m\}$, and the output set $\beta = \{j_1, j_2, \ldots, j_m\}$. The input and output location matrices $\mathbf{E}_\alpha$ and $\mathbf{E}_\beta$ are then defined as $\mathbf{E}_\alpha = [\mathbf{e}_{i_1} \ldots \mathbf{e}_{i_m}]$ and $\mathbf{E}_\beta = [\mathbf{e}_{j_1} \ldots e_{j_m}]$, where $\mathbf{e}_k \in \mathbb{R}^n$ is the canonical unit vector of $\mathbb{R}^n$. It is of course possible to switch between the system description given by (5) and the usual first-order state-space description (1) by taking as state variable $\mathbf{x}' = [\mathbf{q}' \;\; \dot{\mathbf{q}}']$; then, assuming $\mathbf{U}$ to be non-singular,

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{U}^{-1}\mathbf{Z} & -\mathbf{U}^{-1}\mathbf{V} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \mathbf{U}^{-1}\mathbf{E}_\alpha \mathbf{H} \end{bmatrix}, \quad (7)$$
$$\mathbf{C} = [\mathbf{C}_\mathrm{p}\mathbf{E}'_\beta \;\; \mathbf{C}_\mathrm{v}\mathbf{E}'_\beta],$$

where $N = 2n$ is the dimension of the state-space system. Taking the $\mathscr{L}$-transform of both sides, (5) becomes

$$\mathbf{D}(s)\mathbf{q}(s) = \mathbf{E}_\alpha \mathbf{H}\mathbf{u}(s), \quad (8)$$

where $\mathbf{D}(s) = \mathbf{U}s^2 + \mathbf{V}s + \mathbf{Z}$, is the *dynamic matrix* of the system. Under the assumption that $\mathbf{D}$ is invertible, $\mathbf{q}$ is obtained as $\mathbf{q}(s) = \mathbf{D}^{-1}(s)\mathbf{E}_\alpha \mathbf{H}\mathbf{u}(s)$, and, from (6), $\mathbf{y} = \mathbf{C}_0(s)\mathbf{E}'_\beta \mathbf{q}$, where $\mathbf{C}_0(s) = \mathbf{C}_\mathrm{v}(s\mathbf{I} - \tilde{\mathbf{C}})$ and $\tilde{\mathbf{C}} = -\mathbf{C}_\mathrm{v}^{-1}\mathbf{C}_\mathrm{p}$. Then

$$\mathbf{y}(s) = \mathbf{C}_0(s)\mathbf{E}'_\beta \mathbf{D}^{-1}(s)\mathbf{E}_\alpha \mathbf{H}\mathbf{u}(s) = \mathbf{F}(s)\mathbf{u}(s), \quad (9)$$

which characterizes the input–output behaviour of the system. It is easy to show that $|\mathbf{D}(s)| = |\mathbf{U}|\,|s\mathbf{I} - \mathbf{A}| = \gamma p(s)$, where $\gamma = |\mathbf{U}|$ is a non-zero constant. Using the notation

$$\mathbf{D}^{-1}(s) = \frac{1}{|\mathbf{D}(s)|}\mathrm{adj}\,\mathbf{D}(s) = \frac{1}{\gamma p(s)}\mathbf{N}(s),$$

and considering the fact that $\mathbf{E}'_\beta \mathbf{X}\mathbf{E}_\alpha = \mathbf{X}_{\beta,\alpha}$ for any $\mathbf{X} \in \mathbb{C}^{n,n}$, it is easily seen from (9) that

$$\mathbf{F}(s) = \frac{1}{\gamma p(s)}\mathbf{C}_0(s)\mathbf{N}_{\beta,\alpha}(s)\mathbf{H}. \quad (10)$$

On the other hand, the same $\mathbf{F}$ could be derived from the state-space representation of the system:

$$\mathbf{F}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{1}{p(s)}\mathbf{C}\,\mathrm{adj}\,(s\mathbf{I} - \mathbf{A})\mathbf{B}.$$

Comparing this expression with (10), it follows that

$$\mathbf{C}\,\mathrm{adj}\,(s\mathbf{I} - \mathbf{A})\mathbf{B} = \frac{1}{\gamma}\mathbf{C}_0(s)\mathbf{N}_{\beta,\alpha}(s)\mathbf{H}. \quad (11)$$

3.1. *Characterization of the zero polynomial*

Theorem 1. Let $\mathbf{D}(s)$ *be the* $n \times n$ *dynamic matrix of a system described by* (5) *and let* $\mathbf{F}(s)$ *be the transfer-function matrix defined as in* (10). *Under the assumptions of complete controllability and observability, the zero polynomial for the system* (5) *is given by*

$$z(s) = \eta g(s)\,|\mathbf{D}_{\bar{\alpha},\bar{\beta}}(s)|, \quad (12)$$

where η *is a non-zero constant,* $g(s) = \Pi_{k=1}^{m}(s - \lambda_k)$, *and* λ_k, $k = 1, \ldots, m$, *are the eigenvalues of the matrix* $\tilde{\mathbf{C}} = -\mathbf{C}_\mathrm{v}^{-1}\mathbf{C}_p$.

Proof. Recalling the definition of the zero polynomial (4) and using (11),

$$z(s) = \frac{1}{\gamma^m p^{m-1}(s)}|\mathbf{C}_0(s)|\,|\mathbf{N}_{\beta,\alpha}(s)|\,|\mathbf{H}|; \quad (13)$$

now applying (2) to $|\mathbf{N}_{\beta,\alpha}(s)|$ gives

$$|\mathbf{N}_{\beta,\alpha}(s)| = (-1)^\xi\,|\mathbf{D}(s)|^{m-1}\,|\mathbf{D}_{\bar{\alpha},\bar{\beta}}(s)|$$
$$= (-1)^\xi \gamma^{m-1} p^{m-1}(s)\,|\mathbf{D}_{\bar{\alpha},\bar{\beta}}(s)|, \quad (14)$$

which, substituted into (13), gives

$$z(s) = \frac{(-1)^\xi\,|\mathbf{H}|}{\gamma}|\mathbf{C}_0(s)|\,|\mathbf{D}_{\bar{\alpha},\bar{\beta}}(s)|. \quad (15)$$

Considering that $|\mathbf{C}_0(s)| = |\mathbf{C}_\mathrm{v}^{-1}|\,|s\mathbf{I} - \tilde{\mathbf{C}}|$, (12) is easily identified, and the theorem is proved. □

The roots of $z_s(s) = |\mathbf{D}_{\bar{\alpha},\bar{\beta}}(s)|$ will be called *structural zeros*, since they are related to the dynamic matrix of the system and to the positions at which sensors and actuators are located. The roots of $g(s)$ will be called *non-structural zeros*, since they are determined by the output influence matrices $\mathbf{C}_\mathrm{v}$ and $\mathbf{C}_\mathrm{p}$. It should be pointed out that if only the generalized positions $\mathbf{q}$ are measured then the matrix $\mathbf{C}_\mathrm{v}$ does not appear in (6), and the $g(s)$ term of (12) becomes just a multiplicative constant, making no contribution to the TZ of the system. While the $g(s)$ term of (12) could be important in zero design problems, our discussion will in the sequel focus only on the *structural* part of (12), described by

$$z_s(s) = |\mathbf{D}_{\bar{\alpha},\bar{\beta}}(s)|. \quad (16)$$

Equation (16) states that the (structural) transmission zeros of a MIMO second-order system are the values of the complex-frequency s that annihilate the determinant of the submatrix of $\mathbf{D}(s)$ obtained by deleting the *row* indexes α corresponding to the selected *input* locations and the *column* indexes β corresponding to the selected *output* locations.

4. *Structural interpretation of TZ*

The result given in the previous section will now be exploited in order to physically identify the autonomous subsystems of the original system, which are responsible for the zero behaviour. In the colocated case, it is straightforward to identify the zero-providing subsystems, constraining the original system at the sensor/actuator positons. In the non-colocated case, the identification is more involved, and further assumptions are needed on the system. In the latter case, an alternative and promising approach seem to be one based on energetic considerations and bond graphs (Van de Straete, 1996).

4.1. *Colocated systems.* Consider a system in the form (5), with m colocated input and output devices: $\alpha = \beta$ and $\mathbf{E}_\alpha = \mathbf{E}_\beta = \mathbf{E}$. From a physical point of view, a colocated system is defined as one without any energy storage element between the actuator location i_k and the sensor location j_k. The matrix $\mathbf{D}_{\bar{\alpha},\bar{\beta}}$ in (12) is then obtained by deleting the $i_1, \ldots, i_m$ rows and the corresponding columns from matrix $\mathbf{D}(s)$. It is easy to see from (8) that deleting column i_k of $\mathbf{D}(s)$ means forcing the corresponding generalized displacement q_{ik} to zero, i.e. putting a constraint on the i_kth system node. Deleting row i_k of $\mathbf{D}(s)$ means that the equilibrium equation of the i_kth node is no longer taken into account. Deleting simultaneously the i_kth row and column of matrix $\mathbf{D}(s)$, a submatrix is obtained that represents the dynamic matrix of the reduced-order system obtained by constraining to zero the q_{i_k}th coordinate of the structure. The same reasoning applies for all $k = 1, \ldots, m$. The following key result can then be stated.

Proposition 1. *For a second-order MIMO system described by* (5), *with colocated sensor-actuator pairs, the structural transmission zeros are the natural frequencies of the subsystems obtained by constraining to zero the generalized coordinates of the original system at all the input–output locations.*

The illustrated paradigm finds its most natural application in the field of structural control, where system models are often directly provided in the form (5) by a finite element method (FEM) code. In this case, the zeros can be simply and effectively computed as the resonant frequencies of the original structure, with additional constraints imposed on the nodal points corresponding to the sensor and actuators locations.

Considering structural systems of the form (5), with $\mathbf{U} > 0$, $\mathbf{V} = \mathbf{O}$, $\mathbf{Z} \geq 0$ (flexible structures with no damping) and m colocated transducers, it is straightforward to recover the well-known pole–zero interlacing property (Martin and Bryson, 1980), valid for SISO colocated control systems, and to extend it to the MIMO case. Using the Cholesky decomposition $\mathbf{U} = \mathbf{G}\mathbf{G}'$, the dynamic matrix of the system can be written as $\mathbf{D}(s) = \mathbf{G}(\mathbf{I}s^2 + \tilde{\mathbf{Z}})\mathbf{G}'$, where $\tilde{\mathbf{Z}} = \mathbf{G}^{-1}\mathbf{Z}\mathbf{G}'^{-1}$, and the poles of the system are the square roots of the eigenvalues of $-\tilde{\mathbf{Z}}$. According to (16), the (structural) transmission zeros are given by the latent roots of $\mathbf{E}'\mathbf{G}(\mathbf{I}s^2 + \tilde{\mathbf{Z}})\mathbf{G}'\mathbf{E}$, which, if $\mathbf{G}'\mathbf{E} = \mathbf{Q}\mathbf{R}$ is the QR decomposition of $\mathbf{G}'\mathbf{E}$, coincide with the square roots of the eigenvalues of $-\mathbf{Z}_0 = -\mathbf{Q}'\tilde{\mathbf{Z}}\mathbf{Q}$, where $\mathbf{Q}$ has orthonormal columns. The Poincaré separation theorem (Horn and Johnson, 1985) then states that

$$\lambda_i(\tilde{\mathbf{Z}}) \leq \lambda_i(\mathbf{Z}_0) \leq \lambda_{i+m}(\tilde{\mathbf{Z}}), \quad 1 \leq i \leq n - m. \tag{17}$$

The above expression gives bounds for the TZ of a MIMO undamped structure, and clearly reduces to the SISO interlacing property for $m = 1$.

The characterization of TZ given by (16) can also be effectively used to devise an algorithm for zero computation, alternative to the standard methods (Emami-Naeini and Van Dooren, 1982), which avoids computations in the state space, thus halving the size of the involved matrices. For flexible mechanical structures with colocated transducers, if proportional damping is assumed, i.e. if $\mathbf{V}$ is a linear combination of $\mathbf{U}$ and $\mathbf{Z}$, the structural zeros are the latent roots of $\mathbf{D}_0(s) = \mathbf{E}'\mathbf{G}(\mathbf{I}s^2 + \tilde{\mathbf{V}}s + \tilde{\mathbf{Z}})\mathbf{G}'\mathbf{E}$, $\tilde{\mathbf{V}} = \mathbf{G}^{-1}\mathbf{V}\mathbf{G}'^{-1}$ and $\tilde{\mathbf{Z}} = \mathbf{G}^{-1}\mathbf{Z}\mathbf{G}'^{-1}$, and can be computed in the following way. Using the QR decomposition of $\mathbf{G}'\mathbf{E}$ (for details, see Golub and Van Loan, 1989), we have $\mathbf{D}_0(s) = \mathbf{R}'(\mathbf{I}s^2 + \mathbf{V}_0 s + \mathbf{Z}_0)\mathbf{R}$, where $\mathbf{V}_0 = \mathbf{Q}'\tilde{\mathbf{V}}\mathbf{Q}$ and $\mathbf{Z}_0 = \mathbf{Q}'\tilde{\mathbf{Z}}\mathbf{Q}$, and it follows from the proportional damping assumption that $\mathbf{V}_0$ and $\mathbf{Z}_0$ are simultaneously diagonalizable. Then there exists an orthonormal matrix $\mathbf{S} \in \mathbb{R}^{n-m, n-m}$ such that $\mathbf{S}'\mathbf{Z}_0\mathbf{S} = \mathrm{diag}\{z_1^2, \ldots, z_{n-m}^2\}$ and $\mathbf{S}'\mathbf{V}_0\mathbf{S} = \mathrm{diag}\{2\xi_1 z_1, \ldots, 2\xi_{n-m} z_{n-m}\}$, where z_i and ξ_i are the moduli and damping ratios of the structural zeros respectively.

4.2. *Non-colocated systems.* In order to extend the structural interpretation of TZ given in Section 4.1 for colocated systems to non-colocated systems, a restriction of the class of suitable systems is needed. The result that follows is stated without proof, and is valid for systems described by (5), with $\mathbf{U}$ diagonal and $\mathbf{V}$ and $\mathbf{Z}$ tridiagonal, with m actuators at nodes $\alpha = \{i_1, i_2, \ldots, i_m\}$, and m sensors at nodes $\beta = \{j_1, j_2, \ldots, j_m\}$ that interlace along the structure: $i_1 < j_1 < i_2 < j_2 < \ldots < i_m < j_m$.

Proposition 2. *Consider a system described by* (5), *with* $\mathbf{U} \in \mathbb{R}^{n,n}$ *diagonal and* $\mathbf{V}$, $\mathbf{Z} \in \mathbb{R}^{n,n}$ *tridiagonal, and let the actuator and sensor location sets* α *and* β *be such that* $i_1 < j_1 < i_2 < j_2 < \ldots < i_m < j_m$ (*or* $j_1 > i_1 < j_2 < i_2 < \ldots < j_m < i_m$ *respectively*). *The structural transmission zeros can be determined in the following way.*

1. *Constrain to zero the generalized displacements of the system at all the sensors and actuators positions, i.e.* $q_p = 0$, $p \in \{\alpha \cup \beta\}$. *This produces a number of decoupled subsystems of the original system, each delimited by a sensor on the right and an actuator on the left* (*or vice versa respectively*). *Let* t_i *be the number of actuators minus the number of sensors present on the left of node* i.

2. *Let* $S1$ *be the set of subsystems of* (5) *for which* $t_i = 0$ *for all nodes in the subsystem. Let* $S2$ *be the set of subsystems for which* $t_i \neq 0$ *for all the nodes in the subsystem.*

3. *The set of transmission zeros is given by the union of the natural frequencies* (*poles*) *of all the subsystems in* $S1$, *plus the union of the roots of*

$$\prod_{a=i_k}^{j_k-1} (v_{a+1,a}s + z_{a+1,a})$$

$$\left(\text{or } \prod_{a=j_k}^{i_k-1} (v_{a,a+1}s + z_{a,a+1}) \text{ respectively}\right)$$

for each subsystem k *in* $S2$, $k = 1, \ldots, m$, *where* $v_{x,y}$ *and* $z_{x,y}$ *are the entries in position* x, y *of the matrices* $\mathbf{V}$ and $\mathbf{Z}$ respectively.

5. *Conclusions*

The topic of transmission zeros has been analyzed for linear systems described in matrix second-order form. From Theorem 1, the zero polynomial is shown to be composed of two terms: a structural part, related to the system dynamics and transducer locations, and a non-structural part, related to the output influence matrices $\mathbf{C}_p$ and $\mathbf{C}_v$. Section 4 has shown that the structural zeros are the natural frequencies (poles) of subsystems of the original system, and rules for identifying the zero-providing subsystems have been given in Sections 4.1 and 4.2 for the colocated and non-colocated cases respectively. The interpretation of the TZ for the non-colocated case is still restricted to the particular class of systems with tridiagonal $\mathbf{V}$ and $\mathbf{Z}$ structure. Further research in this direction is expected to give results on TZ valid approximately for classes of systems with more practical relevance, such as those showing tridiagonal dominance in the $\mathbf{V}$ and $\mathbf{Z}$ matrix structure.

The proposed approach provides a deep physical and energetic insight in the zero phenomenon, and can also be exploited to devise an efficient algorithm for the computation of the zeros (Section 4.1) for FEM models of colocated flexible structures.

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