Some old and new rationality results for moduli space of curves

Gianfranco Casnati

Dipartimento di Scienze Matematiche – Politecnico di Torino

Dedicated to Andrea on the occasion of his retirement
Ferrara – March 19, 2013
1 Preliminary material
   - Notation
   - My collaboration with Andrea

2 (Unir)rationality of $\mathcal{M}_{g,n}$
   - The very first definitions and results on (uni)rationality
   - Historical overview on unirationality
   - Historical overview on rationality

3 (Unir)rationality of $\mathcal{M}^1_{g,n;d}$
   - The gonality stratification
   - Tetragonal curves
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\begin{itemize}
    \item \(\mathbb{C}\) is the field of complex numbers, \(GL_k\) the general linear group of \(k \times k\) matrices with entries in \(\mathbb{C}\), \(PGL_k\) the projective linear group, i.e. \(GL_k\) modulo the subgroup of scalar matrices.
    \item \(\mathbb{P}^r\) is the projective \(r\)–space over \(\mathbb{C}\).
    \item A curve \(C\) is a projective scheme of dimension 1.
    \item \(\mathcal{M}_{g,n}\) is the coarse moduli space of smooth, projective \(n\)–pointed curves of genus \(g\). We simply write \(\mathcal{M}_g\) instead of \(\mathcal{M}_{g,0}\).
    \item \(\mathcal{M}_{g,n}\) is an irreducible quasi–projective scheme. When \(g \geq 2\) its dimension is \(3g - 3 + n\).
    \item A point of \(\mathcal{M}_{g,n}\) is then an ordered \((n + 1)\)–tuple \((C, p_1, \ldots, p_n)\), where \(C\) is a smooth, projective curve of genus \(g\) and \(p_1, \ldots, p_n \in C\) are pairwise distinct point.
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My scientific collaboration with Andrea Del Centina began in 1994, when I was researcher in Padova, after I got my PhD in Pisa with F. Catanese.

In 1994, I was at the beginning of my scientific activity and I submitted a paper containing, besides some general results, two examples where I claimed the rationality of certain moduli spaces. Unfortunately, the proofs were affected by the standard mistake of many wrong rationality proofs, as the referee pointed out. Essentially, I forgot the action of a finite group.

When I discussed about this fact with my advisor, A.T. Lascu, and with Ph. Ellia they both suggested to ask Andrea who had moved from Urbino to Ferrara as full professor.
Thus I met Andrea and we spoke about such kind of results. In particular he explained to me a very interesting result on bielliptic curve, one of his favorite research topics with F. Bardelli and A. Gimigliano.

I found such a topic interesting. It was a very pleasant mixture of the properties of the canonical embedding, of the geometry of nets of quadrics, of classical and new invariant theory. Then our collaboration started. Since then we published the following nine joint papers.


Besides the interest of the problems which we were dealing with and of the technical details of invariant theory we needed to prove them, a very intriguing ingredient of the proofs in the aforementioned papers was certainly the underlying geometric construction of the curves satisfying the property we were interested in.

Andrea was an expert in such fundamental geometric constructions. Moreover he also was an expert in explaining the geometry behind the problem to me in such a way as to make it easy to handle.
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An algebraic variety $X$ is called unirational if there exists a dominant rational map $\mathbb{P}^r \dashrightarrow X$.

$X$ is called rational if there exists a birational map $\mathbb{P}^r \dashrightarrow X$.

For varieties of dimension 1 and 2 unirationality and rationality are equivalent notions due to well–known theorems by Lüroth and Castelnuovo.


We are interested in dealing with the rationality of $\mathcal{M}_{g,n}$ and of some its subloci of curves satisfying particular properties.
Let $\mathcal{M}$ be one of such loci. The standard approach is to construct a dominant rational map $\phi: V \dashrightarrow \mathcal{M}$ whose fibres are the orbits of a suitable action of a group $G$ on an scheme $V$ which is easy to handle. At this point one proves the rationality of the space $\mathcal{M}$, by proving the rationality of the geometric quotient $V/G$.

Typically $V$ is a suitable subset of the linear system of plane curves of some fixed degree and $G$ is a suitable subgroup of $\text{PGL}_3$. 

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We first recall some facts for $n = 0$. $\mathcal{M}_0$ is a point, $\mathcal{M}_1 \cong \mathbb{A}^1$, the identification being given by the $j$–invariant. Thus $\mathcal{M}_g$ is rational for $g = 0, 1$.

J. Igusa in Ann. of Math. 72 (1960) proved the rationality of $\mathcal{M}_2$ (see also F. Bardelli–A. Del Centina, Math. Ann. 270 (1985), where a different proof resting on geometrical considerations is provided in characteristic 0).

**Question**

Is $\mathcal{M}_g$ rational for all $g \geq 2$?
Using suitable plane models F. Severi was able to prove in Rend. R. Acc. Naz. Lincei 24 (1921) that $\mathcal{M}_g$ is unirational for $g \leq 10$ and conjectured

**Conjecture**

$\mathcal{M}_g$ is unirational for all $g$.

Severi’s proof has been analyzed and demonstrated in a rigorous form by E. Arbarello and E. Sernesi in Duke Math. J. 46 (1979).
\( \mathcal{M}_g \) is irreducible but only quasi–projective. It is then natural to introduce the so called Deligne–Mumford compactification \( \overline{\mathcal{M}}_g \) (when \( g \geq 2 \)). Its points represent isomorphism classes of nodal curves with finite automorphism group.

In the construction of \( \overline{\mathcal{M}}_g \) appears naturally \( \mathcal{M}_{g,n} \) and its compactification via stable \( n \)–pointed curves \( \overline{\mathcal{M}}_{g,n} \). Indeed each component of \( \Delta := \overline{\mathcal{M}}_g \setminus \mathcal{M}_g \) is obtained via identification of pairs of points from moduli spaces of strictly smaller genus.

Thus it also makes sense to compute the Kodaira dimensions \( \kappa_{g,n} \) of \( \overline{\mathcal{M}}_{g,n} \) (or \( \mathcal{M}_{g,n} \)) and \( \kappa_g := \kappa_{g,0} \). Notice that the unirationality of \( \mathcal{M}_{g,n} \) implies \( \kappa_{g,n} = -\infty \).
Theorem

If either \( g \geq 24 \) or \( g = 22 \), then \( \kappa_g = \dim(\overline{M}_g) \). Moreover \( \kappa_{23} \geq 2 \).


The above theorem shows that Severi’s conjecture is false.
The unirationality of $\mathcal{M}_g$ has been proved for $g \leq 14$ in several papers besides the aforementioned ones (for $g = 11, 12, 13$ see M. Chang–Z. Ran, Invent. Math. 76 (1984), for $g = 12$ see E. Sernesi, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 8 (1981), for $g = 14$ see A. Verra, Compos. Math. 141 (2005)).

Though it is known that $\kappa_g = -\infty$ when $g = 15, 16$ (M. Chang–Z. Ran, J. Differential Geom. 24 (1986) and J. Differential Geom. 34 (1991)), no unirationality results are known in the range $15 \leq g \leq 21$. 
It is natural to look at the analogous problems for $\mathcal{M}_{g,n}$.

In *Amer. J. Math. 125 (2003)*, A. Logan proved that $\overline{\mathcal{M}}_{g,n}$ has canonical singularities if $g \geq 4$. Thus each pluricanonical divisor on the open set of curves without non–trivial automorphisms inside $\overline{\mathcal{M}}_{g,n}$ extends to a pluricanonical divisor on the desingularization of $\overline{\mathcal{M}}_{g,n}$.

Such a result yields several bounds which have been improved by A. Bruno–A. Verra in *Projective varieties with unexpected properties*, (2005), G. Farkas in *Amer. J. Math. 131 (2009)*, E. Ballico–____–C. Fontanari in *Forum Math. 21 (2009)*. Combining the above papers with the content of E. Arbarello–M. Cornalba *Math. Ann. 256 (1981)* we obtain the following result.
**Theorem**

If $2 \leq g \leq 21$, then $\mathcal{M}_{g,n}$ is unirational for $n \leq u(g)$, $\kappa_{g,n} \geq 0$ for $n \geq h(g)$ and $\kappa_{g,n} = \dim(\mathcal{M}_{g,n})$ for $n \geq k(g)$ where $u$, $h$ and $k$ are as follow:

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<th>$g$</th>
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<tr>
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<td>$k(g)$</td>
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**Sketch of the proof**

- **Logan’s bound** $u$ is obtained using particular models for general curves of given genus. E.g.:
  - for $g = 2$, curves of bidegree $(2, 3)$ on a smooth quadric surface;
  - for $g = 3, 4, 5, 6$, canonical models;
  - for $g = 7$, curves of degree $7$ with $8$ nodes;
  - for $g = 8, 9$, plane curves of degree $7$ with $13, 12$ nodes;
  - for $g = 11$, hyperplane sections of a $K$–$3$ surface.

- **The bounds** $h$ and $k$ are obtained by computing the canonical divisor as the sum of an ample and an effective divisor.

If $g = 2, 3$, one cannot use Logan’s approach, because the singularities of $\overline{M}_{g,n}$ are not necessarily canonical.
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On the other hand we have some rationality results for low values of $g$ besides the one for $(g, n) = (2, 0)$.

The rationality of $\mathcal{M}_g$ was proved by P.I. Katsylo (see Math. USSR Izv. 25 (1985), Math. USSR-Sb 72 (1992), Comment. Math. Helvetici 71 (1996)) and N.I. Shepherd–Barron (see Invent. Math. 97 (1989) and Compositio Math. 70 (1989)) in the ten years from 1987 till 1996 for $3 \leq g \leq 6$.

All such results are based on the properties of the canonical models of general curves of such genera. Such models are either complete intersection in the canonical space ($3 \leq g \leq 5$) or on a fixed del Pezzo surface of degree 5 ($g = 6$). When $g \geq 7$ there is not such an easy and useful description.
The rationality of $\mathcal{M}_{g,n}$ has been studied by \textit{--C. Fontanari} in J. Lond. Math. Soc. 75 (2007) and by \textit{E. Ballico--C. Fontanari} in Forum Math. 21 (2009).

\textbf{Theorem}

\textit{If $2 \leq g \leq 6$, then $\mathcal{M}_{g,n}$ is rational for $n \leq r(g)$, where $r$ is as follow:}

\begin{tabular}{c|cccccc}
     $g$ & 2 & 3 & 4 & 5 & 6 \\
     \hline
     $r(g)$ & 12 & 14 & 15 & 12 & 8 \\
     $u(g)$ & 12 & 14 & 15 & 12 & 15 \\
     $h(g)$ & ? & ? & 16 & 15 & 16 \\
     $k(g)$ & ? & ? & 17 & 16 & 17 \\
\end{tabular}

\textit{The last three lines being repeated here for reader’s benefit.}
Sketch of the proof

Assume $g \geq 4$. If $(C, p_1, \ldots, p_n)$ is a general pointed curve, then $|K - p_1 - \cdots - p_{g-3}|$ induces a birational map $\varphi: C \to \overline{C} \subseteq \mathbb{P}^2$ coinciding with the projection of its canonical model from the points $p_1, \ldots, p_{g-3}$.

Overline{C} is an integral $(g + 1)$–tic passing through the points $A_i := \varphi(p_i)$ which are in general position in the plane.

If $\ell \subseteq \mathbb{P}^2$ is a line, then $\ell \cdot \overline{C} + A_1 + \cdots + A_{g-3}$ defines a divisor in the canonical system of $C$ by construction.

Moreover its singularities are exactly $\delta$ nodes $N_1, \ldots, N_\delta$ in general position in the plane, where $2g = g(g - 1) - 2\delta$ i.e.

$$\delta = \frac{(g^2 - 3g)}{2}.$$  

It follows the existence of a unique $(g - 3)$–tic $E_C$ through $N_1, \ldots, N_\delta$. 

Sketch of the proof

Since the canonical class is cut out on \( \overline{C} \) by the adjoints of degree \( g - 2 \), then

\[
\ell \cdot \overline{C} + A_1 + \cdots + A_{g-3} \sim \ell \cdot \overline{C} + E_C \cdot \overline{C} - 2N_1 - \cdots - 2N_\delta.
\]

It follows that

\[
E_C \cdot \overline{C} - 2N_1 - \cdots - 2N_\delta \sim A_1 + \cdots + A_{g-3}
\]

If \( C \) is general, then it does not carry \( g_{g-3}^1 \), hence equality holds in the above formula, i.e. \( E_C \) contains \( A_1, \ldots, A_{g-3} \).

Thus we can associate to \( (C, p_1, \ldots, p_{g-3}) \) the curve \( \overline{C} \) with \( \delta \) nodes and \( g - 3 \) distinguished points on a unique \( (g - 3) \)-tic \( E_C \).

If \( g \leq 6 \) the construction above can be reverted. It is easy to prove that two such plane data correspond to the same pointed curve if and only if they are projectively equivalent.
Sketch of the proof

For example let us examine the case $g = 4$. In this case $\overline{C}$ is a plane quintic with two nodes $N_1$, $N_2$ and a distinguished point $A_1$ which are collinear.

Choosing coordinates in $\mathbb{P}^2$ such that $N_1 := [1, 0, 0]$, $N_2 := [0, 1, 0]$, $A_1 := [1, 1, 0]$, then $\overline{C}$ is in a fixed linear system $\Sigma \cong \mathbb{P}^{14}$.

Consider

$$X_n := \left\{ (\overline{C}, A_2, \ldots, A_n) \in \Sigma \times (\mathbb{P}^2)^{n-1} \mid A_i \in \overline{C} \right\}.$$

If $n \leq 15$ the natural projection

$$X_n \to (\mathbb{P}^2)^{n-1}$$

endows $X_n$ with a structure of projective bundle with fibre $\mathbb{P}^{15-n}$.

$\mathcal{M}_{g,n}$ is birational to $X_n/G$, where $G \subseteq \text{PGL}_3$ is the stabilizer of $N_1$ and $N_2$. Its rationality now follows from standard results.
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Every curve of genus $g$ is endowed with a base–point–free $g^1_d$ for each $d \geq \frac{1}{2}g + 1$ thanks to S. Kleiman–D. Laksov, Acta Math. 132 (1974).

Thus it makes sense to consider the loci $M^1_{g,n;d} \subseteq M_{g,n}$ of curves carrying a $g^1_d$ and the gonality stratification

$$M^1_{g,n;2} \subseteq M^1_{g,n;3} \subseteq M^1_{g,n;4} \subseteq \cdots \subseteq M^1_{g,n;\lceil \frac{1}{2}g+1 \rceil} = M_{g,n}.$$

The loci $M^1_{g,n;d}$ are irreducible of dimension $2g + 2d - 5 + n$ if $d \leq \lceil \frac{1}{2}g + 1 \rceil$.

In particular the general point of $M^1_{g,n;d}$ does not lie in $M^1_{g,n;d-1}$. 
It is very easy to prove the rationality of

- $\mathcal{M}_{g,n;2}^1$ with $g \geq 3$ and $1 \leq n \leq 2g + 8$;
- $\mathcal{M}_{g,n;3}^1$ with $g \geq 5$ and $1 \leq n \leq 2g + 7$.


When $n = 0$, analogous results are much more difficult to prove. The rationality of $\mathcal{M}_{g,0;2}^1$ is proved in a long list of papers by P. Katsylo and F. Bogomolov (see Math. USSR Izv. 22 (1984), Math. USSR Izv. 25 (1985), Math. USSR Izv. 25(1985), Math. USSR-Sb. 54 (1986)).

The rationality of $\mathcal{M}_{g,0;3}^1$ is proved in N.I. Shepherd–Barron, Proceedings of Symposia in Pure Mathematics, 46, 1987 when $g \equiv 2 \pmod{4}$ and very recently by S. Ma, arXiv:1012.0983 and arXiv:1207.0184 in the remaining cases.
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The first unknown case is then $\mathcal{M}^1_{g,n;4}$. As we already pointed out such a locus is always unirational (at least when $n = 0$). Thus it is reasonable to deal with its rationality.

The very first case is $g = 7$: thus we deal with the rationality of $\mathcal{M}^1_{7,n;4}$.


**Theorem**

$\mathcal{M}^1_{7,n;4}$ is rational for $n \leq 11$. 

Sketch of the proof

Let $C$ be a tetragonal curve of genus 7 and let $|D|$ a $g^1_4$ on $C$. It is known that if $C$ is general, then such a $g^1_4$ is unique.

In order to prove the statement we distinguish the two cases $n \geq 1$ and $n = 0$.

Sketch of the proof for $n \geq 1$

If $(C, p_1, \ldots, p_n) \in \mathcal{M}^1_{7, n; 4}$ is general, then $|K - D - p_n|$ is a base–point–free $g^2_7$. Let $\varphi : C \to \mathbb{P}^2$ be the corresponding morphism. Let $\overline{C} := \varphi(C)$ and $A_i := \varphi(p_i)$. The curve $\overline{C}$ is a plane septic with 8 nodes $N_1, \ldots, N_8$ which are the base locus of a pencil of cubics, the nineth base point being $A_n$. 
Sketch of the proof for $n \geq 1$

Thus we have a rational map $X_n \dashrightarrow \mathcal{M}^1_{7,n;4}$ defined on the set $X_n$ of $(n + 1)$-tuples of the form

$$(\overline{C}, A_1, \ldots, A_{n-1}, N_1 + \cdots + N_8) \in |\mathcal{O}_{\mathbb{P}^2}(7)| \times (\mathbb{P}^2)^{n-1} \times S^8\mathbb{P}^2$$

where $A_1, \ldots, A_{n-1} \in \overline{C}$, $\overline{C}$ has no other singular points but $N_1, \ldots, N_8$, they are double on $\overline{C}$ and the cubics through the $N_i$'s intersect in another point on $\overline{C}$.

Such a map is actually dominant and equivariant with respect to the natural action of $\text{PGL}_3$. We conclude the proof of the statement in this case proving the rationality of the quotient $X_n/\text{PGL}_3$ when $n \leq 11$ with standard methods.
Sketch of the proof for $n = 0$

The case $n = 0$ needs some more care since we cannot reduced the problem to the analysis of plane curves.

As above let $C$ be a general tetragonal curve of genus 7 and let $|D|$ the unique $g_4^1$ on $C$. Thus $|K - D|$ is a $g_8^3$. If $C$ is general, then one can prove that such a $g_8^3$ is very ample.

Conversely, arguing by symmetry, it is also possible to check that if a curve $C$ of genus 7 is endowed with a very ample $g_8^3$, then it also carries a unique $g_4^1$.

Riemann–Roch theorem yields

$$h^0(\mathbb{P}^3, \mathcal{I}_C(3)) \geq 2$$

for each curve $C$ of degree 8 and genus 7 in $\mathbb{P}^3$. 
Sketch of the proof for $n = 0$

It is easy to check that $C$ is not contained in any quadric surface, thus there are two cubic surfaces $F_1$ and $F_2$ such that $F_1 \cap F_2 = C \cup L$ for a suitable line. Conversely each general curve $C \subseteq \mathbb{P}^3$ linked to a line by a couple of cubic surface, is a general tetragonal curve of genus 7. It follows the existence of a rational dominant map

$$G_2 \dashrightarrow \mathcal{M}_{7,0;4}^1,$$

where $G_2$ denotes the grassmannian of subspaces of dimension 2 of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.

Thanks to the bijection between $g_4^1$’s and $g_8^3$’s on $C$, we obtain that the fibres of the above map are the $\text{PGL}_4$–orbits with respect to the natural action.

The rationality of $\mathcal{M}_{7,0;4}^1$ follows from the rationality of the quotient $G_2/\text{PGL}_4$ and a lot of non–standard representation theory.
The above representation of $C$ as the residual intersection of two cubics with respect to a line has an interesting interpretation. Blowing up the line $L$ we naturally obtain an embedding

$$C \hookrightarrow \mathbb{P} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)).$$

The fibres of the natural projection map $\mathbb{P} \to \mathbb{P}^1$ are planes cutting on $C$ the $g_4^1$. Moreover $C$ is the complete intersection inside $\mathbb{P}$ of two divisors, the proper transforms of the two cubic surfaces.
Such a construction can be generalized to each tetragonal curve $C$ of genus $g \geq 7$. Indeed the divisors $D$ of the $g_4^1$ on $C$ generate a plane and $D$ is always the base locus of a pencil of conics in such a plane. The union of such planes in the canonical space is a scroll $\mathbb{P}$. Moreover $C$ is always the complete intersection of two divisors of $\mathbb{P}$.

Such a description was used in —A. Del Centina in Math. Proc. Cambridge Philos. Soc. 136 (2004) in for proving the rationality of the Weierstrass space of type $(4, g)$, $g \geq 7$, i.e. the locus in $\mathcal{M}_g$ of tetragonal curves with a total ramification point.

In the very recent paper S. Ma in arXiv:1302.3367 such a description has been used as starting point of the proof of the rationality of $\mathcal{M}^1_{g,0;4}$ when $g \equiv 1, 2, 5, 6, 9, 10, \pmod{12}$ and $g \neq 9, 45$. 

G. Casnati (Politecnico di Torino)  
Rationality  
talk @ Ferrara 39 / 39