A Navier-Stokes solver for non-equilibrium aerothermodynamics is coupled with a fully implicit solver of the Maxwell equations. Transport phenomena are modelled using a high accurate Chapman-Enskog method that accounts for the effects of concentration, pressure and temperature gradients and provides anisotropic diffusion velocity, heat flux and viscosity in the presence of a magnetic field. Gas neutrality is not enforced so that the electric charge density can be locally different from zero. Numerical tests are carried out on a flat-faced cylinder to test the capabilities of the implicit Maxwell solver and of the coupled Maxwell/Navier-Stokes system.

I. Introduction

The effects of the interaction between electromagnetic fields and partially ionized gases are interesting for atmospheric reentry applications because they can potentially reduce the heat flux to a space vehicle surface and allow for aerodynamic control. Electromagneto-fluid dynamics is governed by the coupled set of the Maxwell and Navier-Stokes equations. Starting from this complete model, different simplifications based on order-of-magnitude considerations are frequently made to obtain a less complex set of governing equations. In the last years, D’Ambrosio and Giordano considered the possibility of solving the full set of the Maxwell and Navier-Stokes equations in a coupled fashion. This resulted in a series of papers that demonstrated that solving the coupled system is feasible with a reasonable expense of computer processing time and memory.1–5 More recently, the same approach was embraced by MacCormack, who also solved the coupled Maxwell and Navier-Stokes equations in paper presented at the last AIAA Plasmadynamics Conference.6 The full set of electromagneto-fluid dynamics equations is an extremely powerful investigation tool that accounts for effects that are considered negligible a priori by the approximated models. One of its merits is that it can be used to verify when and where the simplifying assumptions on which the reduced models are based are valid, and when and where they are not.

Despite the completeness of the electromagnetic model, an important element was missing in the studies presented in Refs. 1–6, as it is in most numerical simulations of electromagneto-fluid dynamics interactions. The transport model and the definitions of electrical conductivity and conduction current density were not consistent with each other and the plasma was forced to be neutral. In particular, in Refs. 1–5, an arbitrarily constant electrical conductivity was used in connection with the generalized Ohm law to test the numerical solution techniques and the effect of the presence of a magnetic field was not included in the evaluation of transport coefficients. However, as pointed out by Giordano in Ref. 7, the definition of the conduction current density directly descends from the definition of the mass diffusive fluxes $J_{m_i}$ of electrically charged species, to that the direct application of the generalized Ohm law is not necessary. In addition, since in the first-order Chapman-Enskog theory the components mass diffusive flux depends not only on the generalized electric field $E + v \times B$, but on partial pressure gradients and temperature gradients also, there are contributions to electric currents due to pressure and thermal diffusion of charged species also.
Note that an explicit computation of the electrical conductivity is not needed for computational purposes. If desired, presso-electrical, thermoelectrical and electrical conductivities tensors could be post-processed starting from diffusion coefficients. Finally, the presence of a magnetic field influences transport phenomena, which become anisotropic, as reported by Bruno et alii in Refs. 8, 9. Components mass diffusive fluxes and heat fluxes arise in directions parallel and normal to the magnetic field, and they are not necessarily aligned with diffusion driving forces and temperature gradients. The viscosity coefficient is a five-components vector. Part of these phenomena are usually incorporated in the so-called Hall effect, which is frequently accounted for in approximated magneto-fluid dynamics models in the form of a tensorial electrical conductivity. In this paper, the Hall effect will be consistently linked with the components mass diffusive fluxes. Last, but not the least, the plasma will not be forced to be neutral, so that an electric charge density $\rho_c$ and the related convection current density $\rho_c v$ may be present in the flowfield.

The possible presence of a convection current and the anisotropy of transport coefficients in the presence of a magnetic field has a strong impact on the resulting flowfields. At a microscopic level, all originates from the dynamics of particles carrying an electric charge $q$ moving in a magnetic field $B$ with speed $v$ and thus subject to the Lorentz force $q v \times B$. At the macroscopic levels, this results in forces that are due to convection currents $\rho_c v$ and conduction currents $J_Q$. Consider for example an axially symmetric configuration, as those that are frequently adopted in plasma wind tunnels to carry out experiments on the electromagneto-fluid dynamics interaction. One consequence of the plasma non-neutrality and, possibly to a much larger extent, of transport coefficients anisotropy is that such flows, which are axisymmetric in absence of a magnetic field, will develop a swirling motion in the presence of an applied axisymmetric magnetic field. In addition, an induced magnetic field component will appear in the azimuthal direction and electric fields and electric currents will have components in the meridian plane also.

In this paper, we will show numerical results obtained simulating the flowfield about a blunt-faced cylinder in a plasma consisting of argon atoms, singly charged ions, and electrons. The geometrical, fluid dynamics and magnetic configuration is similar to the one that was recently set up in the experiments by Gulhan et alii at DLR-Koeln. The fully coupled Maxwell and Navier-Stokes equations will be solved as in Ref. 5 with the addition of the complete transport model described in Refs. 8, 11. The axisymmetric EMFD equations will include the momentum equation in the azimuthal direction, the azimuthal velocity component and the projection of the Maxwell equations in every coordinate direction.

## II. Full magneto-fluid dynamics equations

### II.A. Maxwell equations

The Maxwell equations read like

\[
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0
\]

(1)

\[
\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \varepsilon_0 c^2 \nabla \times \mathbf{B} = -\mathbf{j} = -\rho_c v - \mathbf{J}_Q
\]

(2)

The electric current density, $\mathbf{j}$, represents the flux of electric charge due to convection (convection current) and diffusion (conduction current):

\[
\mathbf{j} = \rho_c v + \mathbf{J}_Q
\]

(3)

Electric charge density and conduction current density can be obtained directly from the species partial densities and from the mass-diffusive fluxes that appear in the governing equations of fluid dynamics:

\[
\rho_c = e N_A \sum_{i=1}^{N_{spe}} \frac{\rho_i}{M_i} \sigma_{is}
\]

(4)

\[
\mathbf{J}_Q = e N_A \sum_{i=1}^{N_{spe}} \frac{\mathbf{J}_{mi}}{M_i} \sigma_{is}
\]

(5)

The summation above is extended to all species, $\sigma_{is}$ is +1 for ions, -1 for electrons and 0 for neutral species, $e$ is the electric charge, $J_{mi}$ is the mass-diffusive flux of species $i$ and $\rho_i$ is the species partial density.

The numerical solution of the Maxwell equations can be improved by an appropriate normalization. Here we report four possible forms, all sharing the reference variables $L_{ref}$ for lengths, $V_{ref}$ for velocities and $B_{ref}$ for the magnetic field components:
\( t_{\text{ref}} = \frac{L_{\text{ref}}}{V_{\text{ref}}} \), \( E_{\text{ref}} = V_{\text{ref}} B_{\text{ref}} \), \( J_{\text{Q,ref}} = \lambda_{\text{ref}} V_{\text{ref}} B_{\text{ref}} \)

\[
\frac{\partial B}{\partial t} + \nabla \times E = 0
\]  
\( \text{(6a)} \)

\[
\frac{\partial E}{\partial t} - \hat{c}^2 \nabla \times B = -\rho_c v - \hat{c}^2 (Re_m) V_{\text{ref}} J_Q
\]  
\( \text{(6b)} \)

b) \( t_{\text{ref}} = \frac{L_{\text{ref}}}{V_{\text{ref}}} \), \( E_{\text{ref}} = c B_{\text{ref}} \), \( J_{\text{Q,ref}} = \lambda_{\text{ref}} c B_{\text{ref}} \)

\[
\frac{\partial B}{\partial t} + \hat{c} \nabla \times E = 0
\]  
\( \text{(7a)} \)

\[
\frac{\partial E}{\partial t} - \hat{c} \nabla \times B = -\rho_c v - \hat{c} (Re_m) V_{\text{ref}} J_Q
\]  
\( \text{(7b)} \)

c) \( t_{\text{ref}} = \frac{L_{\text{ref}}}{V_{\text{ref}}} \), \( E_{\text{ref}} = c B_{\text{ref}} \), \( J_{\text{Q,ref}} = \lambda_{\text{ref}} V_{\text{ref}} B_{\text{ref}} \)

\[
\frac{\partial B}{\partial t} + \nabla \times E = 0
\]  
\( \text{(8a)} \)

\[
\frac{\partial E}{\partial t} - \hat{c} \nabla \times B = -\rho_c v - \hat{c} (Re_m) V_{\text{ref}} J_Q
\]  
\( \text{(8b)} \)

d) \( t_{\text{ref}} = \frac{L_{\text{ref}}}{c} \), \( E_{\text{ref}} = c B_{\text{ref}} \), \( J_{\text{Q,ref}} = \lambda_{\text{ref}} V_{\text{ref}} B_{\text{ref}} \)

\[
\frac{\partial B}{\partial t} + \nabla \times E = 0
\]  
\( \text{(9a)} \)

\[
\frac{\partial E}{\partial t} - \nabla \times B = -\frac{1}{\hat{c}} \rho_c v - \hat{c} (Re_m) V_{\text{ref}} J_Q
\]  
\( \text{(9b)} \)

where

\[
\hat{c} = \frac{c}{V_{\text{ref}}}
\]  
\( \text{(10a)} \)

\[
(Re_m) V_{\text{ref}} = \frac{\lambda_{\text{ref}} V_{\text{ref}} L_{\text{ref}}}{\varepsilon_0 \hat{c}^2}
\]  
\( \text{(10b)} \)

Forms b) and c) provide the best numerical results because the elements of the Jacobian of the electromagnetic fluxes are of the same order of magnitude. Thus, form c) will be used in the remaining part of this paper.

The two Gauss’ laws for magnetostatics and electrostatics, that directly derive from the Maxwell equations, must always be numerically satisfied:

\[
\nabla \cdot B = 0
\]  
\( \text{(11)} \)

\[
\varepsilon_0 \nabla \cdot E = \rho_c
\]  
\( \text{(12)} \)

Equation (12) can be used to define the reference electric charge density

\[
\rho_{\text{ref}} = \varepsilon_0 \frac{E_{\text{ref}}}{L_{\text{ref}}}
\]  
\( \text{(13)} \)

and, in nondimensional form, it will read as

\[
\nabla \cdot E = \rho_c
\]  
\( \text{(14)} \)

In the numerical method that we propose here, we will integrate the Maxwell equations using a finite volumes method. To this purpose, the integral form of the equations will be used:

\[
\int_V \frac{\partial B}{\partial t} dV + \hat{c} \int_S (\mathbf{n} \times \mathbf{E}) dS = 0
\]  
\( \text{(15)} \)

\[
\int_V \frac{\partial E}{\partial t} dV - \hat{c} \int_S (\mathbf{n} \times \mathbf{B}) dS = -\hat{c} \int_V \left( \frac{\rho_c v}{\hat{c}} + Re_m J_Q \right) dV
\]  
\( \text{(16)} \)
II.B. Navier-Stokes equations

The Navier-Stokes equations for a non-equilibrium chemically reacting gas are linked to the Maxwell equations through source terms that represent Lorentz force and Joule heating. In integral and non-dimensional form they read like:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{17a}
\]

\[
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{vv} + p \mathbf{I}) - \frac{\sqrt{\gamma M}}{Re} \nabla \cdot \mathbf{\tau} = S \rho \mathbf{E} + S \left( \frac{\rho_0 \mathbf{v}}{c} + Re_m \mathbf{J}_Q \right) \times \mathbf{B} \tag{17b}
\]

\[
\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] + \frac{\sqrt{\gamma M}}{Re} \nabla \cdot (\mathbf{J}_e - \mathbf{\boldsymbol{\tau}} \cdot \mathbf{v}) = \dot{c} S \left( \frac{\rho_0 \mathbf{v}}{c} + Re_m \mathbf{J}_Q \right) \cdot \mathbf{E} \tag{17c}
\]

\[
\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{v}) = \Omega_i \tag{17d}
\]

The non-dimensional parameter \( S \) is called the magnetic force number and it is defined as:

\[
S = \frac{\varepsilon_0 c^2 B^2}{\rho_0 V_{el}^2} \tag{18}
\]

The product \( S Re_m \) gives non-dimensional parameter \( Q \), which is called the magnetic interaction parameter and it is defined as:

\[
Q = \lambda_{el} B^2 L \rho_0 V_{el} \tag{19}
\]

The total energy per unit volume, \( E \), is given by

\[
E = \sum_{i=1}^{n} \rho_i e_i + \rho \frac{\|\mathbf{v}\|^2}{2} \tag{20}
\]

Symbol \( J_i \), represents the flux of energy due to transport phenomena. As noticed before, the conduction current density, \( J_Q \), can be computed as described in Eq. (5) without the need of invoking the generalized Ohm law. To this purpose, transport phenomena in a ionized gas in the presence of an electromagnetic field should be properly modelled.

II.C. Transport model

The transport model that is used in this work is described in Refs. 8, 11 and is based on the Chapman-Enskog method. A noticeable consequence of the presence of a magnetic field is that transport properties cease to be isotropic, but they depend on the orientation of the magnetic field. A new frame of reference with components parallel (∥), perpendicular (⊥) and tangent (t) to the magnetic field vector must be defined. To move from cartesian components, \( \mathbf{v}^c = \{v_x, v_y, v_z\} \), to rotated components, \( \mathbf{v}^r = \{v^\parallel, v^\perp, v^t\} \), a rotation matrix \([R]\) must be defined:

\[
\mathbf{v}^r = [R] \mathbf{v}^c \tag{21}
\]

\[
[R] = \begin{bmatrix}
  b_x & b_y & b_z \\
  -b_y & b_x + \frac{b_y^2}{1 + b_x} & -\frac{b_z}{1 + b_x} \\
  -b_z & \frac{b_y b_z}{1 + b_x} & b_x + \frac{b_y^2}{1 + b_x}
\end{bmatrix} \tag{22}
\]

where

\[
b_i = \frac{B_i}{|B|} \tag{23}
\]

Diffusion velocity and heat flux components in the directions parallel, perpendicular and tangential to the magnetic field can be obtained solving algebraic systems. For the parallel component we have:

\[
\alpha [A] \{x^\parallel\} = \{b^\parallel\} \tag{24}
\]

and for the normal and tangential components:

\[
(\alpha [A] + \iota \beta [B]) \{x^\perp + \iota x^t\} = \{b^\perp + \iota b^t\} \tag{25}
\]
Matrices $[A]$ and $[B]$ contain elements that are related to collisional cross sections. Their rank depends upon the number of species and upon the order of approximation of the Chapman-Enskog method. The right-hand side contains the diffusion driving forces and the temperature gradients:

\[
\left\{ b^i \right\} = \begin{bmatrix}
  \vdots \\
  \frac{5}{2} x_i \left( \nabla T \right)^{i} \\
  -d_i^{\parallel} \\
  -d_{i+1}^{\parallel} \\
  \vdots \\
  \frac{5}{2} x_{i+1} \left( \nabla T \right)^{i} \\
  0 \\
  0 \\
  \vdots 
\end{bmatrix} \\
\left\{ b^i + i b^f \right\} = \begin{bmatrix}
  \vdots \\
  \frac{5}{2} x_i \left( \nabla T \right)^{i} \\
  \left( \nabla T \right)^{i} + i \left( \nabla T \right)^{f} \\
  \vdots \\
  \frac{5}{2} x_{i+1} \left( \nabla T \right)^{i} \\
  \left( \nabla T \right)^{i} + i \left( \nabla T \right)^{f} \\
  \vdots 
\end{bmatrix}
\] (26)

with

\[
d_i^{\parallel} = \left( \nabla x_i \right)^{\parallel} + \left( x_i - y_i \right) \left( \nabla \ln p \right)^{\parallel} - \frac{x_i}{k_B T} \left( \rho c_i - m_i \frac{\rho c}{\rho} \right) \left[ E^{\parallel} + (\mathbf{v} \times \mathbf{B})^{\parallel} \right]
\] (27a)

\[
d_i^{\perp} = \left( \nabla x_i \right)^{\perp} + \left( x_i - y_i \right) \left( \nabla \ln p \right)^{\perp} - \frac{x_i}{k_B T} \left( \rho c_i - m_i \frac{\rho c}{\rho} \right) \left[ E^{\perp} + (\mathbf{v} \times \mathbf{B})^{\perp} \right]
\] (27b)

\[
d_i^{t} = \left( \nabla x_i \right)^{t} + \left( x_i - y_i \right) \left( \nabla \ln p \right)^{t} - \frac{x_i}{k_B T} \left( \rho c_i - m_i \frac{\rho c}{\rho} \right) \left[ E^{t} + (\mathbf{v} \times \mathbf{B})^{t} \right]
\] (27c)

The unknowns are the diffusion velocity ($J_{m_i} = \rho_i w_i$) and the heat flux:

\[
\left\{ x^{\parallel} \right\} = \begin{bmatrix}
  \vdots \\
  x_i w_i^{\parallel} \\
  x_{i+1} w_{i+1}^{\parallel} \\
  \vdots \\
  \vdots \\
  \text{h.o.t.} \\
  \text{h.o.t.} \\
  \vdots
\end{bmatrix} \\
\left\{ x^{\perp} + i x^{t} \right\} = \begin{bmatrix}
  \vdots \\
  x_i w_i^{\perp} + i (x_i w_i^{t}) \\
  x_{i+1} w_{i+1}^{\perp} + i (x_{i+1} w_{i+1}^{t}) \\
  \vdots \\
  \vdots \\
  \text{h.o.t.} \\
  \text{h.o.t.} \\
  \vdots
\end{bmatrix}
\] (28)

\[
q^{\parallel} = \frac{5}{2} p \sum_{i=1}^{n} q_i^{\parallel} \\
q^{\perp} = \frac{5}{2} p \sum_{i=1}^{n} q_i^{\perp} \\
q^{t} = \frac{5}{2} p \sum_{i=1}^{n} q_i^{t}
\] (29)

**II.D. Axis-symmetric finite-volumes form**

Here, we will restrict our attention to axisymmetric fields. This means that changes in the axial $x$-directions and radial $r$-direction will be allowed, but changes in the azimuthal $\theta$-direction will be assumed to be impossible. However, no restrictions are made on the azimuthal components of vector fields. In the finite volumes discretization, the equation system to be solved for each computational cell is:

\[
\frac{\partial \rho}{\partial t} \mathbf{A} + \sum_{i=1}^{4} \left[ \rho (u_{x_i} n_x + u_{r_i} n_r) \right] \cdot \hat{\mathbf{A}} = 0
\] (30)
\[
\begin{align*}
\frac{\partial (\rho u_x)}{\partial t} \mathbf{A} \mathbf{r} + & \sum_{i=1}^{4} \left[ \rho u_x (u_x n_x + u_r n_r) \right]_i \ell_i \bar{r}_i + \sum_{i=1}^{4} (p n_x)_i \ell_i \bar{r}_i - \sum_{i=1}^{4} (\tau_{xx} n_x + \tau_{rx} n_r)_i \ell_i \bar{r}_i = \\
= & S \rho_e E_x \mathbf{A} \mathbf{r} + S (j_x B_\theta - j_\theta B_x) \mathbf{A} \mathbf{r} \\
\frac{\partial (\rho u_r)}{\partial t} \mathbf{A} \mathbf{r} + & \sum_{i=1}^{4} \left[ \rho u_r (u_x n_x + u_r n_r) \right]_i \ell_i \bar{r}_i + \sum_{i=1}^{4} (p n_r)_i \ell_i \bar{r}_i - \sum_{i=1}^{4} (\tau_{xx} n_x + \tau_{rx} n_r)_i \ell_i \bar{r}_i + \\
= & S \rho_e E_x \mathbf{A} \mathbf{r} + S (j_x B_e - j_\theta B_x) \mathbf{A} \mathbf{r} \\
\frac{\partial (\rho u_\theta)}{\partial t} \mathbf{A} \mathbf{r} + & \sum_{i=1}^{4} \left[ \rho u_\theta (u_x n_x + u_r n_r) \right]_i \ell_i \bar{r}_i - \sum_{i=1}^{4} (\tau_{x\theta} n_x + \tau_{r\theta} n_r)_i \ell_i \bar{r}_i = \\
= & S \rho_e E_\theta \mathbf{A} \mathbf{r} + S (j_x B_\theta - j_\theta B_e) \mathbf{A} \mathbf{r} \\
\frac{\partial E_x}{\partial t} \mathbf{A} \mathbf{r} + & \sum_{i=1}^{4} \left[ (E + p) (u_x n_x + u_r n_r) \right]_i \ell_i \bar{r}_i + \sum_{i=1}^{4} (J_{U_x} n_x + J_{U_r} n_r)_i \ell_i \bar{r}_i - \\
= & \frac{e c}{2} (j_x E_x + j_r E_r + j_\theta E_\theta) \mathbf{A} \mathbf{r} \\
\frac{\partial \rho_j}{\partial t} \mathbf{A} \mathbf{r} + & \sum_{i=1}^{4} \left[ \rho_j (u_x n_x + u_r n_r) \right]_i \ell_i \bar{r}_i + \sum_{i=1}^{4} \left[ (J_{M_j})_x n_x + (J_{M_j})_r n_r \right]_i \ell_i \bar{r}_i = \Omega_j \mathbf{A} \mathbf{r} \\
& \text{for } j = 1, N_{spe} \\
\frac{\partial B_x}{\partial t} \mathbf{A} \mathbf{r} + & \frac{e c}{2} \left[ \sum_{i=1}^{4} (E_\theta n_x)_i \ell_i \bar{r}_i + (E_x)_i \right] = 0 \\
\frac{\partial B_\theta}{\partial t} \mathbf{A} \mathbf{r} + & \frac{e c}{2} \sum_{i=1}^{4} (-E_\theta n_x)_i \ell_i \bar{r}_i = 0 \\
\frac{\partial E_x}{\partial t} \mathbf{A} \mathbf{r} - & \frac{e c}{2} \left[ \sum_{i=1}^{4} (B_\theta n_x)_i \ell_i \bar{r}_i + (B_x)_i \right] = - \frac{e c}{2} j_x \mathbf{A} \mathbf{r} \\
\frac{\partial E_\theta}{\partial t} \mathbf{A} \mathbf{r} - & \frac{e c}{2} \sum_{i=1}^{4} (-B_\theta n_x)_i \ell_i \bar{r}_i = - \frac{e c}{2} j_\theta \mathbf{A} \mathbf{r}
\end{align*}
\]
III. Numerical solution of the Maxwell equations

III.A. Full implicit integration of the Maxwell equations

In the work described in this paper the Maxwell equations are integrated using a fully implicit scheme:

\[ \mathbf{J}^m \Delta \mathbf{W}_{be} = -\mathbf{F}^m - \mathbf{S}^m A\bar{r} \]  

(42)

where

\[ \mathbf{W}_{be} = \{ B_x, B_r, B_\theta, E_x, E_r, E_\theta \}^T \]  

(43)

and

\[ \mathbf{F}^m = \begin{cases} \hat{c} \left[ \sum_{i=1}^{4} (E_\theta n_r)_i \xi_i \bar{r}_i \right] \\ \hat{c} \sum_{i=1}^{4} (-E_\theta n_x)_i \xi_i \bar{r}_i \\ \hat{c} \left[ \sum_{i=1}^{4} (E_r n_x - E_x n_r)_i \xi_i \bar{r}_i \right] \\ -\hat{c} \sum_{i=1}^{4} (B_\theta n_r)_i \xi_i \bar{r}_i + (B_\theta)_i A \bar{r}_i \\ -\hat{c} \sum_{i=1}^{4} (-B_\theta n_x)_i \xi_i \bar{r}_i = -\hat{c} J_r A\bar{r}_r \\ \hat{c} \left[ \sum_{i=1}^{4} (B_r n_x - B_x n_r)_i \xi_i \bar{r}_i \right] \end{cases} \]  

(44)

\[ \mathbf{S}^m = \{ 0, 0, 0, j_x, j_r, j_\theta \}^T \]  

(45)

The Jacobian elements \( \frac{\partial \mathbf{S}^m}{\partial \mathbf{W}_{be}} \) are obtained through numerical differentiation of the electric current density components with respect to the electromagnetic field components. The Jacobian elements \( \frac{\partial \mathbf{F}^m}{\partial \mathbf{W}_{be}} \) are computed analytically. Their form depends on the numerical schemes that is used to evaluate electromagnetic fluxes at cells interfaces, a subject that will be addressed to in the next subsection.

III.B. A flux-difference splitting method for the Maxwell equations

The integration in time of the Maxwell equations using a finite volumes technique requires the evaluation of the integrals \( \int_S (\mathbf{n} \times \mathbf{E}) \, dS \) and \( \int_S (\mathbf{n} \times \mathbf{B}) \, dS \). This means that one has to compute the tangential components of \( \mathbf{E} \) and \( \mathbf{B} \) at each computational cell surface. To obtain a stable explicit numerical method, a flux-difference splitting technique is used. The technique is very similar to the one that we use for the flow solver\(^\text{12} \) and consists in solving a one dimensional Riemann problem normal to each cell surface. We define the locally one-dimensional Riemann problem and its solution as follows.

III.B.1. Locally one-dimensional Riemann problem

Riemann invariants and propagation speeds:

\[ dR^+_{te} = dE_z + dB_t \quad ; \quad \lambda = -\hat{c} \]  

(47)

\[ dR^+_{te} = dE_z - dB_t \quad ; \quad \lambda = +\hat{c} \]  

(48)

\[ dR^-_{tm} = dE_t - dB_z \quad ; \quad \lambda = -\hat{c} \]  

(49)

\[ dR^+_{tm} = dE_t + dB_z \quad ; \quad \lambda = +\hat{c} \]  

(50)

Indicating with 'l' and 'r' left and right states respectively, and with '*' the intermediate state, after integration of the Riemann invariants, the Riemann problem solution is:

\[ (R^+_{te})^r = E^*_z + B^*_t \]  

(51)

\[ (R^+_{te})^l = E^*_z - B^*_t \]  

(52)

\[ (R^-_{tm})^r = E^*_t - B^*_z \]  

(53)

\[ (R^+_{tm})^l = E^*_t + B^*_z \]  

(54)
Using the Riemann solver described above, the electromagnetic fluxes at cells interfaces are given as follows:

\[
B_t^* = \frac{(R_{\text{rb}})^r - (R_{\text{te}})^l}{2} = \frac{B_t^l + B_t^r}{2} - \frac{E_z^l - E_z^r}{2}
\]

\[
E_t^* = \frac{(R_{\text{te}})^l + (R_{\text{me}})^r}{2} = \frac{E_t^l + E_t^r}{2} + \frac{B_z^l - B_z^r}{2}
\]

\[
B_z^* = \frac{(R_{\text{te}})^l - (R_{\text{me}})^r}{2} = \frac{B_z^l + B_z^r}{2} + \frac{E_t^l - E_t^r}{2}
\]

\[
E_z^* = \frac{(R_{\text{rb}})^r + (R_{\text{te}})^l}{2} = \frac{E_z^l + E_z^r}{2} - \frac{B_t^l - B_t^r}{2}
\]

Subscripts \(T-E\) and \(T-M\) stand for transverse electric and transverse magnetic, the two propagation modes in which the signal can be decomposed. In transverse electric propagation mode, components \(E_z\), \(E_y\) and \(B_z\) are perturbed, while in the transverse magnetic mode the signal is a combination of the \(B_z\), \(E_x\) and \(E_y\) components.

### III.B.2. The half Riemann problem at the boundaries

In case only the magnetic or the electric field are specified at the boundaries, Riemann invariants can be used to evaluate what is missing. In particular, we have:

a) \(B_t\) is known, \(E_z\) is unknown:

\[
E_{z}^{bc} = E_{z}^{in} - (B_{t}^{in} - B_{t}^{bc})
\]  

b) \(E_z\) is known, \(B_t\) is unknown:

\[
B_{t}^{bc} = B_{t}^{in} - (E_{z}^{in} - E_{z}^{bc})
\]

c) \(E_t\) is known, \(B_z\) is unknown:

\[
B_{z}^{bc} = B_{z}^{in} + (E_{t}^{in} - E_{t}^{bc})
\]

d) \(B_z\) is known, \(E_t\) is unknown:

\[
E_{t}^{bc} = E_{t}^{in} + (B_{z}^{in} - B_{z}^{bc})
\]

### III.B.3. Maxwell fluxes Jacobian

Using the Riemann solver described above, the electromagnetic fluxes at cells interfaces are given as follows:

\[
\mathbf{F}^m = \hat{c} \begin{pmatrix}
    n_y \left( \frac{E_t^l + E_t^r}{2} - \frac{-n_y B_x^l + n_x B_y^l + n_y B_x^r - n_x B_y^r}{2} \right) \\
    -n_x \left( \frac{E_t^l + E_t^r}{2} - \frac{-n_y B_x^l + n_x B_y^l + n_y B_x^r - n_x B_y^r}{2} \right) \\
    -n_y \left( \frac{B_z^l + B_z^r}{2} + \frac{-n_y E_x^l + n_x E_x^l + n_y E_x^r - n_x E_x^r}{2} \right) \\
    n_x \left( \frac{B_z^l + B_z^r}{2} + \frac{-n_y E_x^l + n_x E_x^l + n_y E_x^r - n_x E_x^r}{2} \right) \\
    n_y \left( \frac{B_z^l + B_z^r}{2} + \frac{-n_y E_x^l + n_x E_x^l + n_y E_x^r - n_x E_x^r}{2} \right) \\
    -n_x \left( \frac{B_z^l + B_z^r}{2} + \frac{-n_y E_x^l + n_x E_x^l + n_y E_x^r - n_x E_x^r}{2} \right)
\end{pmatrix}
\]  

\[
= \hat{c} \begin{pmatrix}
    n_y E_z^* \\
    -n_x E_z^* \\
    E_t^* \\
    -n_y B_z^* \\
    n_x B_z^* \\
    -B_t^*
\end{pmatrix}
\]
Therefore, the dependence of the electromagnetic fluxes with respect to 'left' and 'right' cells is easily obtained:

\[
\left( \frac{\partial F^{\text{em}}}{\partial \mathbf{w}_{\text{nc}}} \right)_i = \frac{1}{2} \mathbf{\ell}_i \mathbf{\bar{r}}_i
\]

\[
\left( \frac{\partial F^{\text{em}}}{\partial \mathbf{w}_{\text{nc}}} \right)_i = \frac{1}{2} \mathbf{\ell}_i \mathbf{\bar{r}}_i
\]

These elements of the Jacobian matrix can be computed just once at the beginning of the numerical simulation. Conversely, the complete Jacobian must be computed at each time step because of the non-linear dependence of the electric current density from the electromagnetic field components and because the integration time step may change from step to step.

### III.C. Enforcing \( \nabla \cdot \mathbf{B} = 0 \) and \( \nabla \cdot \mathbf{E} = \rho_c / \varepsilon_0 \)

The Gauss’ laws for magnetostatics and electrostatics defined in Eqs. (11) and (12) may not be satisfied by the numerical solution of the Maxwell equations due to discretization errors. Especially when the magnitude of the computed electromagnetic field is large, it is necessary to enforce the two conditions independently from the integration of the Maxwell equations. To this purpose, we use a projection method similar to the one described in Ref. 13. The numerical solutions of the electromagnetic field, \( \mathbf{B}^n \) and \( \mathbf{E}^n \), are corrected by a quantity that can be defined as the gradient of two scalar functions, \( \psi_b \) and \( \psi_k \), respectively.

\[
\mathbf{B} = \mathbf{B}^n + \nabla \psi_b \quad (67a)
\]

\[
\mathbf{E} = \mathbf{E}^n + \nabla \psi_k \quad (67b)
\]

The needed correction can be computed by applying the divergence operator to equations Eqs. (67a) and (67b) and using Eqs. (11) and (14):

\[
\nabla \cdot \mathbf{B}^n + \nabla \cdot \nabla \psi_b = 0 \quad (68)
\]

\[
\nabla \cdot \mathbf{E}^n + \nabla \cdot \nabla \psi_k = \rho_c \quad (69)
\]

In practice, a Poisson equation has to be solved for each Gauss’ law. Ignoring the numerical error in evaluating \( \rho_c \), the two equations will read like:

\[
\nabla \cdot \nabla \psi_b = -\nabla \cdot \mathbf{B}^n \quad (70)
\]

\[
\nabla \cdot \nabla \psi_k = \rho_c^n - \nabla \cdot \mathbf{E}^n \quad (71)
\]

The projection method will also be affected by the spatial accuracy of the Poisson solver.

### IV. Numerical solution of the Navier-Stokes equations

The Navier-Stokes equations are discretized according to a finite volumes method based on the same grid where the Maxwell equations are computed. Convective fluxes are evaluated using an upwind flux-difference splitting technique\(^{12}\) and diffusive fluxes through a centered scheme. Second order accuracy in space and in time is achieved using an Essentially Non-Oscillatory scheme. Chemical source terms are treated implicitly to avoid instabilities due to fast chemical reactions. The volumetric force \( \mathbf{j} \times \mathbf{B} \) and heating \( \mathbf{j} \cdot \mathbf{E} \) due to the electromagnetic field are also treated implicitly, at least as far as fluid dynamics variables are concerned. To this purpose, the Jacobian of the electromagnetic source terms is computed through numerical differentiation and then summed to the Jacobian of chemical source terms, so that a tight coupling between fluid dynamics and chemical and electromagnetic source terms is enforced.
V. Coupling technique for Maxwell and Navier-Stokes equations

Maxwell and Navier-Stokes equations are loosely coupled here. For each time step, we solve first the Navier-Stokes equations while keeping the electromagnetic field frozen. Then, we integrate the Maxwell equations using the same time step of fluid dynamics or using a series of sub-steps until the same simulation time of fluid dynamics is reached. During the Maxwell time step, fluid dynamics variables are assumed to be frozen. As an attempt to mitigate the unstable behavior that will be discussed later, a predictor-corrector scheme was also attempted. It consists in making one full fluid dynamics step, then one Maxwell step using the new fluid dynamics variables, and finally in recovering the initial fluid dynamics variables and in marching the Navier-Stokes equations forward in time again with the updated electromagnetic field values. Indicating with $\mathcal{F}$ and $\mathcal{M}$ the Navier-Stokes and Maxwell operators, respectively, this can be summarized as

$$W_{\nu}^{K+1},* = \mathcal{F}(W_{\nu}^K, W_{\nu}^{BE}) \quad \text{explicit N.-S.} \quad (72a)$$

$$W_{BE}^{K+1} = \mathcal{M}(W_{\nu}^{K+1},*, W_{BE}^K, W_{BE}^{*}) \quad \text{implicit Maxwell} \quad (72b)$$

$$W_{\nu}^{K+1,*} = \mathcal{F}(W_{\nu}^K, W_{BE}^{*}) \quad \text{explicit N.-S.} \quad (72c)$$

VI. Numerical results

VI.A. Geometrical configuration and computational grid

Numerical experiments have been carried out about a flat-faced cylinder that contains a coil arranged as in Fig. 1. Such a model is very similar to configuration C3 described in Ref. 10.

![Figure 1. Meridian section of the flat-faced cylindrical model including current carrying wires.](image)

The computational domain for both Maxwell and Navier-Stokes equations have been discretized using the mesh shown in Fig. 2. The Maxwell equations are solved everywhere, including the interior of the model. The computational domain must be extended far from the model with respect to typical CFD simulations because the electromagnetic field will affect regions outside the shock layer.

![Figure 2. Computational mesh (left). Enlargement in the vicinity of the model (right).](image)

VI.B. Applied magnetic field reconstruction using the implicit Maxwell solver

As a first numerical experiment, we demonstrate the capability of the implicit Maxwell solver to compute the applied magnetic field. A similar results was shown in Ref. 5 using a different numerical technique and on a slightly different
configuration. The capability of autonomously computing the applied magnetic field is interesting because it avoids the necessity of interpolating an externally computed magnetic field in our computational domain. In order to compute the applied magnetic field using the Maxwell equations, we need to specify $B$ and $E$ of the boundaries of the computational domain. In this case, the boundaries are the freestream, the wires surfaces and the symmetry axis. At the freestream, we assume the magnetic field to be zero and we evaluate the electric field using a half Riemann problem. At the wires surfaces, we specify the magnetic field that we obtain by integrating the generalized Biot-Savart-Laplace law defined in Eq. (73)

$$B(r) = \frac{\mu_0}{4\pi} \iiint_V \frac{j(r') \times (r - r')}{|r - r'|^3} dV$$

(73)

For axially symmetric configurations, there is an analytical solution to Eq. (73) that contains elliptic integrals, which are evaluated numerically.

The solution obtained solving the Maxwell equations with the implicit scheme described above is compared in Fig. 3 with the solution of the generalized Biot-Savart-Laplace law (Eq. (73)). The agreement is quite satisfactory. Enforcing the $\nabla \cdot B = 0$ constraint was mandatory to obtain such a good result.

**VI.C. Electrically non-neutral hypersonic ionizing Argon flow without applied magnetic field**

In order to test the coupling between the Maxwell and Navier-Stokes solver, we run a test case on the flat-faced cylindrical model described above in absence of an applied magnetic field. The gas mixture is pre-ionized Argon. The freestream conditions are listed in Table 1 and they are similar to those measured in a recent experiment by Guelan et al.\textsuperscript{10} The wall is assumed to be non-catalyic and radiation-adiabatic, with emissivity coefficient equal to 0.85. The Maxwell equations are solved in the flowfield and in the solid body.

<table>
<thead>
<tr>
<th>$V_\infty$ [m/s]</th>
<th>$T_\infty$ [K]</th>
<th>$p_\infty$ [Pa]</th>
<th>$(x_e)_\infty$</th>
<th>$(x_{Ar^+})_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2350</td>
<td>80</td>
<td>10</td>
<td>$6.98 \cdot 10^{-5}$</td>
<td>$6.98 \cdot 10^{-5}$</td>
</tr>
</tbody>
</table>

Table 1. Freestream conditions

The challenge in this simulation is that electrical neutrality is not enforced. The combined action of transport and electromagnetic field will distribute charged and neutral particles according to the physical laws embedded in the Maxwell and Navier-Stokes equations. Since no magnetic field is applied in the meridian plane, transport properties will remain isotropic and we expect to see an electric field in the meridian plane and magnetic field in the azimuthal
plane. The initial solution is the flowfield obtained in the same conditions, but assuming electric neutrality and ambipolar diffusion. The initial electric field is the one that comes out assuming that conduction currents are zero in the flowfield.

We tried to run our computation using different time steps. If the fluid dynamics time step is used from the beginning (CFL$_{\text{CFD}} = 8 \cdot 10^{-1}$, CFL$_{\text{Maxw}} = 1.39 \cdot 10^{5}$), the CFD solution blows up immediately and the Maxwell step cannot even start. To obtain a solution that remains stable for a reasonable number of steps, we have to reduce the CFL of 1000 times at least. Using a CFL$_{\text{CFD}} = 8 \cdot 10^{-4}$, that is CFL$_{\text{Maxw}} = 139$, the number of iterations is sufficiently large to provide a meaningful solution. In particular, it is clearly possible to see a loss of neutrality across the shock wave and at the wall. The electric charge density and the axial component of the electric field along the stagnation line are shown in Fig. 4. As we expect from the electrostatic Gauss law, $E_x$ decreases when the electric charge density is negative and increases when $\rho_c$ is positive. The non-neutrality is due to an accumulation of electrons before the shock wave and a reduction of electrons after the shock. We interpret this fact noting that for light particles it is harder to cross the strong adverse pressure gradient with respect to heavy particles. A smaller deviation from neutrality, in this
case with an excess of positively charged particles, can also be seen at the wall. Electric current densities distributions along the stagnation line are shown in Fig. 5. The conduction current, \( J_{Q} \), is the major component of the total electric current density \( j \).

Despite these results are quite encouraging and unique, the numerical solution does not remain stable for a long time. Instabilities start to occur in various spots across the shock wave and we were not able to keep them under control up to now. The situation is depicted in Fig. 6, where electric charge density and electric field magnitude contours are plotted and electric field lines are superimposed. The spots that are located near the shock are regions of exceedingly high electric charge density and electric field magnitude that cannot be taken under control and continue to increase until the numerical simulation blows up. The same behavior occurs using smaller time steps also and the instability appears to arise roughly at the same physical time regardless of the used time step.

![Figure 6. Electric charge density contours and electric field lines (left). Magnitude of the electric field and electric field lines (right).](image)

The source of instability doesn’t seem to be linked to the solution of the Maxwell equations, but rather to the drift term that is embedded in the transport model implemented in the Navier-Stokes equations. The solution to this problem will be the subject of future research.

**VII. Conclusions**

In this paper we discussed four elements of our research work that can be considered as new in the framework of the numerical modelling and simulation of magneto-fluid dynamics. First, we described a fully implicit numerical method for the Maxwell equations and we successfully used it to evaluate the magnetostatic field generated by a coil in flat-faced cylinder model. Second, we successfully implemented accurate transport models based on the Chapman-Enskog method in our magneto-fluid dynamics solver. Third, we coupled our implicit Maxwell solver with our explicit non-equilibrium Navier-Stokes solver. Fourth, we did not enforce gas neutrality, thus allowing the electric charge density to varying in the flowfield according to the governing equations. The latter point resulted to be critical, because strong instabilities arise after a certain amount of time in those regions where the electric charge density differ from zero. We suspect that such a problem is linked with the drift term in the transport model, but we have not found a method to keep the instability under control yet.

**References**


