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Hamiltonian formulation for surface waves in a layered fluid

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Abstract

An Hamiltonian formulation for irrotational isoentropic flow of two fluids of different density subject to the gravity force is given. Canonical variables are the elevation of the free surface and the jump of the momentum potential density evaluated at the interface between fluids and at the upper free boundary.

1. Introduction

A large class of water waves at the free surface of a fluid of constant density can be studied assuming irrotational isoentropic flow [1,2]. The Laplace equation, holding in the interior of the fluid, must be supplemented by two nonlinear coupled equations at the free surface: one describing the evolution in time of the free surface itself, the other ensuring the boundary condition for the elliptic problem. In this context, the dynamics of surface waves of a non-stratified fluid can be described in any regime of wave height or length until they do not break down.

Because of the assumption of potential velocity, it is not surprising that the dynamics of the waves is completely determined by quantities evaluated at the boundary of the fluid, namely the elevation of the free-surface and the velocity potential evaluated at the free surface itself. The former fixes the fluid domain, the latter provides boundary conditions for a well posed elliptic boundary value problem of mixed type. Much less obviously, Zakharov [3] showed that the water elevation and the potential at the free surface are canonical variables when formulating the water-waves problem in an Hamiltonian formalism, the Hamilton functional being the total energy of the fluid. This result was pioneered by Luke [4] who showed that the boundary conditions of free surface flow can be obtained by a variational formulation.

The mathematical properties of the Hamiltonian formalism for free-surface waves has then been extensively studied by Miles [5], Milder [6], Radder [7] and many other authors. They have highlighted that incompressibility and irrotationality constitute a set of ideal constraints on the motion of the fluid which can then be described in terms of Lagrangian coordinates. In this context, the forces operating in the interior of the domain to satisfy the mechanical constraints (the *reaction* of the fluid to the constraints) can be ignored. This makes the existence of Bernoulli's integral of motion in the interior not needed for the description of the wave motion.

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Hamiltonian formulation of inviscid free-surface flow is not restricted at all to incompressible irrotational fluids. These assumptions can be relaxed, and the constraints of mass conservation and free surface moving with the normal velocity of the fluid enter into the Lagrangian functional thanks to Lagrange multipliers, the classical approach of Hamiltonian fluid dynamics [8–10]. In such a framework the flow cannot be any more described in terms of quantities defined at the free surface only.

In this paper a problem which is somehow intermediate between the two scenarios outlined above is addressed. It is shown that the Hamiltonian formalism holding for a constant-density fluid can be extended to a system composed by two irrotational fluids with different density subject to gravity. Moreover, the result can be even generalized to *N*-layers.

Waves at the interface between immiscible fluids occur in several technological applications [11] as well as environmental flows: for example, temperature and salinity gradients of the oceanic water body often yield to a very sharp density front that the wave dynamics at the termohaline is well represented by a two layers model.

The Hamiltonian formulation of internal waves is possible after clearly stating the dynamic boundary conditions holding at the interface between the fluids (i.e. the balance of surface forces) in terms of water elevation and potential. It is shown that canonical variables for the Hamiltonian description are the water elevation at the upper free surface and at the surface separating the two fluids, and the gap in momentum potential density computed at the surfaces themselves.

2. Primitive-equations formulation

We consider two fluids with constant density ρ_1 and ρ_2, subject to the gravity force in a horizontally unbounded domain. The surface $z = \zeta_1(x, y, t) = \zeta_1(\mathbf{x}, t)$ (the *interface*) separates the two fluids and the surface $z = \zeta_2(\mathbf{x}, t)$ (the *free surface*) separates the upper fluid from the air, supposed to be at constant pressure $p = 0$. For the sake of simplicity, the lower fluid is supposed to be infinitely deep. At the far field the water is at rest: $\zeta_1(+\infty) = 0$ and $\zeta_2(+\infty) = h = \text{const}$, and therein the (hydrostatic) pressure at the interface is $p_0 = \rho_2 g h$.

The velocity potential of the lower and upper fluid are

$$\phi_1(\mathbf{x}, z, t), \quad -\infty \le z \le \zeta_1, \tag{1}$$

$$\phi_2(\mathbf{x}, z, t), \quad \zeta_1 \le z \le \zeta_2. \tag{2}$$

The velocity potentials satisfy the Laplace equation:

$$\frac{\partial^2 \phi_1}{\partial z^2} + \nabla^2 \phi_1 = 0, \tag{3}$$

$$\frac{\partial^2 \phi_2}{\partial z^2} + \nabla^2 \phi_2 = 0, \tag{4}$$

where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^{\mathrm{T}}$. In order to ensure that the system has finite energy (see below) all the unknowns are supposed to have integrable square derivatives in a bounded set and vanish out of it. The upper and lower surfaces are defined by the relationships

$$f_1(\mathbf{x}, z, t) = z - \zeta_1(\mathbf{x}, t) = 0, \tag{5}$$

$$f_2(\mathbf{x}, z, t) = z - \zeta_2(\mathbf{x}, t) = 0. \tag{6}$$

In general, at $z = \zeta_1$ the velocity potential of the fluids are not equal, but the velocity of the fluids normal to the separating surface must be equal. This condition reads (see [2], p. 433):

$$\frac{\mathrm{d} f_1}{\mathrm{d} t} = 0, \qquad \frac{\mathrm{d} f_2}{\mathrm{d} t} = 0, \tag{7}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ indicates the total derivative. Eqs. (7) in explicit form read

$$\frac{\partial \zeta_1}{\partial t} = \frac{\partial \phi_1}{\partial z} - \nabla\zeta_1 \cdot \nabla\phi_1 = \frac{\partial \phi_2}{\partial z} - \nabla\zeta_1 \cdot \nabla\phi_2, \quad \text{for } z = \zeta_1, \tag{8}$$

$$\frac{\partial \zeta_2}{\partial t} = \frac{\partial \phi_2}{\partial z} - \nabla\zeta_2 \cdot \nabla\phi_2, \quad \text{for } z = \zeta_2. \tag{9}$$

Eqs. (8) and (9) (*kinematical conditions*) say that the fluid at the interface moves with the velocity of the interface itself. They provide the motion of the free surface depending on the flowfield.

The pressure gap across the interface and free surface is equal to the force per unit surface exerted by the surface tension (see [12], p. 217):

$$p_1(\mathbf{x}, z = \zeta_1) - p_2(\mathbf{x}, z = \zeta_1) = -\tau_1 \nabla \cdot \left(\frac{\nabla\zeta_1}{\sqrt{1 + (\nabla\zeta_1)^2}} \right), \tag{10}$$

$$p_2(\mathbf{x}, z = \zeta_2) = -\tau_2 \nabla \cdot \left(\frac{\nabla\zeta_2}{\sqrt{1 + (\nabla\zeta_2)^2}} \right), \tag{11}$$

where τ_1 and τ_2 are the surface tension coefficients corresponding to the interfaces ζ_1 and ζ_2, respectively.

The Bernoulli integrals in each fluid read:

$$\frac{p_1}{\rho_1} + \frac{\partial \phi_1}{\partial t} + \frac{1}{2}\left[(\nabla\phi_1)^2 + \left(\frac{\partial \phi_1}{\partial z} \right)^2 \right] + gz = \frac{p_0}{\rho_1}, \tag{12}$$

$$\frac{p_2}{\rho_2} + \frac{\partial \phi_2}{\partial t} + \frac{1}{2}\left[(\nabla\phi_2)^2 + \left(\frac{\partial \phi_2}{\partial z} \right)^2 \right] + gz = \frac{p_0}{\rho_2}, \tag{13}$$

where g is the acceleration of gravity. At the right-hand side of Eqs. (12) and (13) the same reference pressure appears, because the two fluids are assumed to be at rest at infinity: p_0 is the common far-field value of the pressure at the interface, where the surface tension vanishes together with the curvature of the surface (as given by the Eqs. (10) and (11)). The pressure of the upper fluid at infinity evaluated using Eq. (12) at $z = \zeta_2$ is zero, coinciding with the constant air pressure. Evaluating Eqs. (12) and (13) at $z = \zeta_1$ and $z = \zeta_2$, subtracting the former from the latter and using the relationship (10) and (11) we get the *dynamical conditions*:

$$\begin{aligned} &\rho_1 \left(\frac{\partial \phi_1}{\partial t} + \frac{1}{2}\left[(\nabla\phi_1)^2 + \left(\frac{\partial \phi_1}{\partial z} \right)^2 \right] + g\zeta_1 \right) - \tau_1 \nabla \cdot \left(\frac{\nabla\zeta_1}{\sqrt{1 + (\nabla\zeta_1)^2}} \right) \\ &\quad = \rho_2 \left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\left[(\nabla\phi_2)^2 + \left(\frac{\partial \phi_2}{\partial z} \right)^2 \right] + g\zeta_1 \right), \quad \text{for } z = \zeta_1, \end{aligned} \tag{14}$$

$$\rho_2 \left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\left[(\nabla\phi_2)^2 + \left(\frac{\partial \phi_2}{\partial z} \right)^2 \right] + g\zeta_2 \right) - \tau_2 \nabla \cdot \left(\frac{\nabla\zeta_2}{\sqrt{1 + (\nabla\zeta_2)^2}} \right) = \quad p_0 \quad \text{for } z = \zeta_2. \tag{15}$$

3. Hamiltonian formulation

In the previous section a statement of the two-fluids problem has been given in a classical fluid dynamics framework. In the present section we restate the problem in a different way: we consider a mechanical system characterized by a specific kinetic and potential energy, subject to the constraints (3),(4),(8) and (9). It is shown that rewriting the total energy in terms of quantities evaluated at the surfaces, the two-fluids problem has Hamiltonian structure, the Hamilton functional being the total energy itself. The conjugate momentum is nothing but the gap of velocity potential times density at the interfaces and the corresponding Hamilton equations are the dynamical conditions (14) and (15).

The kinetic energy of the two-fluids system is

$$T = \int \mathrm{d}\mathbf{x} \left(\int_{-\infty}^{\zeta_1} \mathrm{d}z \, \frac{\rho_1}{2} \left[(\nabla\phi_1)^2 + \left(\frac{\partial \phi_1}{\partial z} \right)^2 \right] + \int_{\zeta_1}^{\zeta_2} \mathrm{d}z \, \frac{\rho_2}{2} \left[(\nabla\phi_2)^2 + \left(\frac{\partial \phi_2}{\partial z} \right)^2 \right] \right), \tag{16}$$

The potential energy of the two-fluids systems under consideration is infinite, because of the assumption of infinitely deep water. However, we actually are interested in the variations of the potential energy with respect to a reference state. In particular, when considering the variation with respect to the configuration at rest, which holds at infinity ($\zeta_1 = 0$, $\zeta_2 = h$) we get:

$$\begin{aligned} V = \int \mathrm{d}\mathbf{x} \Bigg(& \int_0^{\zeta_1} \mathrm{d}z \int_0^z \mathrm{d}z' \, \rho_1 g + \int_{\zeta_1}^{\zeta_2 - h} \mathrm{d}z \int_0^z \mathrm{d}z' \, \rho_2 g + \tau_1 \left(\sqrt{1 + (\nabla\zeta_1)^2} - 1 \right) \\ & + \tau_2 \left(\sqrt{1 + (\nabla\zeta_2)^2} - 1 \right) \Bigg). \end{aligned} \tag{17}$$

In order to express the Hamilton functional of the system in a simpler form, we wish to rewrite the kinetic and potential energy in terms of quantities *defined* at the surfaces only. Such an aim can be immediately and naturally achieved for the potential energy, but is impossible for the kinetic energy. See [13,14] for a discussion about this issue. However, using integration by parts, it is possible to rewrite the kinetic energy in terms of velocity potential and its derivatives *evaluated* at the surfaces.

The first integral on the right-hand side of Eq. (16) can be rewritten as

$$\begin{aligned} & \int \mathrm{d}\mathbf{x} \int_{-\infty}^{\zeta_1} \mathrm{d}z \, \frac{\rho_1}{2} \left[(\nabla\phi_1)^2 + (\frac{\partial \phi_1}{\partial z})^2 \right] \\ & = \frac{\rho_1}{2} \int \mathrm{d}\mathbf{x} \left(\int_{-\infty}^{\zeta_1} \mathrm{d}z \left[\nabla \cdot (\phi_1 \nabla\phi_1) - \phi_1 \nabla^2 \phi_1 - \phi_1 \frac{\partial^2 \phi_1}{\partial z^2} \right] + \left[\phi_1 \frac{\partial \phi_1}{\partial z} \right]_{z=\zeta_1} \right) \\ & = \frac{\rho_1}{2} \int \mathrm{d}\mathbf{x} \left(\nabla \cdot \int_{-\infty}^{\zeta_1} \mathrm{d}z (\phi_1 \nabla\phi_1) + \left[-\phi_1 \nabla\phi_1 \cdot \nabla\zeta_1 + \phi_1 \frac{\partial \phi_1}{\partial z} \right]_{z=\zeta_1} \right) \\ & = \frac{\rho_1}{2} \int \mathrm{d}\mathbf{x} \left(\phi_1 \left[\frac{\partial \phi_1}{\partial z} - \nabla\phi_1 \cdot \nabla\zeta_1 \right] \right)_{z=\zeta_1}, \end{aligned} \tag{18}$$

where Eq. (3) and the assumption of vanishing velocity at infinity have been used. By carrying out analogous calculations for the second term on the right-hand side of (18), the kinetic energy of the system can be rewritten as

$$\begin{aligned} T = \frac{1}{2} \int \mathrm{d}\mathbf{x} \Bigg(& \rho_1 \left(\phi_1 \left[\frac{\partial \phi_1}{\partial z} - \nabla\phi_1 \cdot \nabla\zeta_1 \right] \right)_{z=\zeta_1} + \rho_2 \left(\phi_2 \left[\frac{\partial \phi_2}{\partial z} - \nabla\phi_2 \cdot \nabla\zeta_2 \right] \right)_{z=\zeta_2} \\ & - \rho_2 \left(\phi_2 \left[\frac{\partial \phi_2}{\partial z} - \nabla\phi_2 \cdot \nabla\zeta_1 \right] \right)_{z=\zeta_1} \Bigg). \end{aligned} \tag{19}$$

Recalling that the normal component of the velocity of the two fluids at the interface must be equal, as expressed by the second relationship of (8), Eq. (9) simplifies to

$$T = \frac{1}{2}\int \mathrm{d}\mathbf{x}\left(\left((\rho_1\phi_1 - \rho_2\phi_2)\left[\frac{\partial\phi_1}{z} - \nabla\phi_1\cdot\nabla\zeta_1\right]\right)_{z=\zeta_1} + \left(\rho_2\phi_2\left[\frac{\partial\phi_2}{\partial z} - \nabla\phi_2\cdot\nabla\zeta_2\right]\right)_{z=\zeta_2}\right). \tag{20}$$

Analogous calculations allow to rewrite the potential energy V in the following form:

$$V = \int \mathrm{d}\mathbf{x}\left[\frac{g}{2}(\zeta_1)^2(\rho_1-\rho_2) + \rho_2\frac{g}{2}(\zeta_2-h)^2 - \tau_1\left(\sqrt{1+(\nabla\zeta_1)^2}-1\right) - \tau_2\left(\sqrt{1+(\nabla\zeta_2)^2}-1\right)\right]. \tag{21}$$

In order to restate the problem in terms of unknowns evaluated on the surfaces only, we define the potential evaluated at surfaces ζ_1 and ζ_2 as the restrictions

$$\varphi_1(\mathbf{x},t) = \phi_1(\mathbf{x}, z=\zeta_1, t), \tag{22}$$

$$\varphi_2(\mathbf{x},t) = \phi_2(\mathbf{x}, z=\zeta_1, t), \tag{23}$$

$$\psi(\mathbf{x},t) = \phi_2(\mathbf{x}, z=\zeta_2, t). \tag{24}$$

Now consider the Dirichlet–Neumann operators G_1 and G_2 as follows [15]. Fix ζ_1, ζ_2, ϕ_1, ϕ_2 and ψ; let φ_1 and φ_2 be the solutions of the boundary value problems

$$\nabla^2\phi_1 = 0, \quad \phi_1 = \varphi_1 \ \text{ for } z=\zeta_1, \tag{25}$$

and

$$\nabla^2\phi_2 = 0, \quad \begin{array}{l}\phi_2 = \varphi_2 \ \text{ for } \ z=\zeta_1,\\ \phi_2 = \psi \ \text{ for } \ z=\zeta_2.\end{array} \tag{26}$$

Then define

$$G_1(\zeta_1)\phi_i = \left(\nabla\varphi_i\cdot\nabla\zeta_1 - \frac{\partial\varphi_i}{\partial z}\right)_{z=\zeta_1} = \sqrt{1+(\nabla\zeta_1)^2}\left(\frac{\partial\varphi_i}{\partial n_1}\right)_{z=\zeta_1}, \tag{27}$$

$$G_2(\zeta_2)\psi = \left(\nabla\varphi_2\cdot\nabla\zeta_2 - \frac{\partial\varphi_2}{\partial z}\right)_{z=\zeta_2} = \sqrt{1+(\nabla\zeta_2)^2}\left(\frac{\partial\varphi_2}{\partial n_2}\right)_{z=\zeta_2}, \quad \text{for } \ i=1,2, \tag{28}$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are the outgoing unitary vectors normal to the boundary of the domains 1 and 2, respectively.

The operators G_1 and G_2 are non-local, they act linearly on the potentials, but depend non-linearly on ζ_1, ζ_2 [6,15]. Then the kinetic energy, in terms of G_1 and G_2 reads:

$$T = \frac{1}{2}\int \mathrm{d}\mathbf{x}\left(\rho_1\varphi_1 G_1\varphi_1 - \rho_2\varphi_2 G_1\varphi_2 + \rho_2\psi G_2\psi\right). \tag{29}$$

A careful reading of definitions (27) and (28) shows that it is impossible to express T explicitly in terms of the surface variables $\zeta_1, \zeta_2, \varphi_1, \varphi_2, \psi$ only.

If we consider the Hamilton functional given by

$$H = T + V, \tag{30}$$

we can define $Q_1 = \zeta_1$ and $Q_2 = \zeta_2$, and the conjugate momenta naturally are

$$P_1 = \rho_1\varphi_1 - \rho_2\varphi_2, \tag{31}$$

$$P_2 = \rho_2\psi. \tag{32}$$

The quantities P_1 and P_2 are momentum potential densities. The functional derivative [16] of the Hamilton functional in terms of the new variables yields to Hamilton equations in canonical form:

$$\frac{\partial Q_1}{\partial t} = \frac{\delta H}{\delta P_1}, \qquad \frac{\partial P_1}{\partial t} = \frac{\delta H}{\delta Q_1}, \tag{33}$$

$$\frac{\partial Q_2}{\partial t} = \frac{\delta H}{\delta P_2}, \qquad \frac{\partial P_2}{\partial t} = -\frac{\delta H}{\delta Q_2}, \tag{34}$$

that coincide with the classical fluid dynamical formulation (8) and (9) and (14) and (15).

Note that the kinematical conditions appearing in Eqs. (33) and (34) are actually derived from the Hamiltonian functional not assumed. Only the geometrical condition that the normal velocity of the fluids at the interface is equal has been used when writing Eq. (20) and a final result is that it is the velocity of the interface itself. Conversely, when considering rotational flow, the kinematical condition has to be appended to the Lagrangian as a constraint by a Lagrange multiplier [8].

4. Conclusive remarks

Hamiltonian functional and canonical variables for gravity waves in a layered free-surface fluid have been derived. The description of the dynamics of a fluid system as an Hamiltonian field offers a number of advantages: the conservation laws and invariances of the theory are in one-to-one relation. This property is conserved when deriving approximate dynamical equations because the approximation, which is applied directly to the fluid Lagrangian, do not disturb the corresponding symmetry properties [17].

The results obtained above can be easily extended to the case of uneven bottom as well as to a system composed by N layers of fluids with different density. Analogously, the upper free surface can be removed, then considering flow in ducts.

The present contribution can be interpreted as a generalization of the usual Hamiltonian formulation for free surface waves to more than one layer. In fact, when $\rho_2 \to 0$ (so that $p_0 \to 0$ and $\zeta_2 \to \zeta_1$) the well known description for one fluid is recovered.

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