

Interaction of two quasi-monochromatic waves in shallow water

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We study the nonlinear interaction of waves propagating in the same direction in shallow water characterized by a double-peaked power spectrum. The starting point is the prototypical equation for weakly nonlinear unidirectional waves in shallow water, i.e., the Korteweg–de Vries equation. In the framework of envelope equations, using a multiple-scale technique and under the hypothesis of narrow-banded spectra, a system of two coupled nonlinear Schrödinger equations is derived. The validity of the resulting model and the stability of their plane wave solutions is discussed. We show that when retaining higher order dispersive terms in the system, plane wave solutions become modulationally unstable. © 2003 American Institute of Physics. [DOI: 10.1063/1.1622394]

The propagation of multiple wave-train systems in shallow water has historically received less attention than the propagation of one single wave train. Nevertheless, experimental studies carried out by Thompson¹ in representative sites near the coasts of the United States reveal that in 65% of the analyzed data, ocean wave spectra show two or more separated peaks in the frequency domain.^{1,2} In this framework, experimental work in the laboratory has been performed by Smith and Vincent.² They propagated irregular wave trains with two distinct spectral peaks in a wave flume with 1:30 slope for different values of the peak frequency and significant wave height. From a physical point of view, this condition mimics the interaction of two wave regimes, a “swell” and a “sea,” propagating in the same direction toward shore in shallow water. Their major observation was a decay of the higher frequency peak along the flume. More recently, using a higher order Boussinesq model, Chen *et al.*³ have shown that nonlinear interactions, without invoking bottom friction or wave breaking, are sufficient to account for the decay of the high frequency peak. Even though these numerical simulations of the Boussinesq equation qualitatively reproduce the experimental results, the basic physical mechanisms of interaction of wave trains with double peaked spectra in shallow water are far from being completely understood.

In this Brief Communication we investigate from a fundamental point of view the possible sources of instability that may occur when two quasi-monochromatic waves interact. Our starting point is the Korteweg–de Vries (KdV) equation, the basic weakly nonlinear model for unidirectional shallow water waves. It is a well-known result that from the KdV equation, in the limit of narrow-banded spectra, the *defocusing* nonlinear Schrödinger (NLS) can be recovered⁴ (see also Ref. 5). The same result can be obtained directly from the Euler equation.⁶ While the *focusing* NLS equation, derived for deep water waves, exhibits the modulational instability,

plane wave solutions of the *defocusing* NLS are stable to side band perturbations. A fundamental question that naturally arises is the following: may the presence of a second peak in the spectrum in shallow water invoke the onset of a new type of instability? In order to address this question we derive from the KdV equation a system of two *defocusing* coupled nonlinear Schrödinger equations (CNLS) and study the stability of its plane wave solutions. Systems of NLS equations are not new in various fields of physics. For example, focusing CNLS equations have been derived^{7,8} and in general can be found for nonlinear media for two waves with different polarizations. For particular values of the coefficients in the CNLS equations, it has also been shown that the system is integrable.^{9,10} Concerning water waves, focusing CNLS equations have been derived by Roskes¹¹ in infinite water depth. CNLS systems describing the interaction of two counter-propagating waves are also discussed in Refs. 12 and 13.

It is well known that the KdV equation can be formally derived from the Euler equations for water waves^{14,15} under the assumption that waves are small (but finite amplitude) and long when compared with the water depth at rest. In a frame of reference moving with the velocity $c_0 = \sqrt{gh}$, where h is the water depth and g is gravity acceleration, the KdV equation in nondimensional form reads

$$\eta_t + \mu \eta \eta_x + \lambda \eta_{xxx} = 0. \quad (1)$$

Here $\eta = \eta(x, t)$ is the free surface elevation, x and t are space and time variables; μ and λ are the nonlinear and dispersive small parameters: $\mu = 3a/2h$ and $\lambda = (h/l)^2/6$, with a a characteristic wave amplitude and l a characteristic wavelength. We are interested in investigating the interaction of two waves, centered at the nondimensional wavenumbers k_1 and k_2 , propagating in the positive x direction. We

consider the case of narrow-banded spectra, i.e., $\Delta k_i/k_i \ll 1$, with $i=1,2$, Δk_i being the characteristic width of spectra around each peak. The approach used below to obtain CNLS equations resembles the one used by Grimshaw *et al.*¹⁶ to derive a single higher order NLS equation starting from an extended KdV (cubic nonlinearity is included). We introduce the following slow space and time variables $X = \epsilon x$ and $T = \epsilon t$, with $\epsilon \ll 1$, and perform a formal expansion of $\eta = \eta(x, t, X, T)$:

$$\eta = \frac{\epsilon}{2}(Ae^{i\Theta_1} + Be^{i\Theta_2}) + \frac{\epsilon^2}{2}(A_2e^{i2\Theta_1} + B_2e^{i2\Theta_2}) + C_2e^{i(\Theta_1+\Theta_2)} + D_2e^{i(\Theta_1-\Theta_2)} + M + \frac{\epsilon^3}{2}(A_3e^{i3\Theta_1} + B_3e^{i3\Theta_2} + C_3e^{i(2\Theta_1+\Theta_2)} + D_3e^{i(2\Theta_1-\Theta_2)} + F_3e^{i(\Theta_1+2\Theta_2)} + G_3e^{i(\Theta_1-2\Theta_2)}) + c.c., \quad (2)$$

where $\Theta_i = k_i x - \omega_i t$ and $A, A_2, A_3, B, B_2, B_3, C_2, C_3, D_2, D_3, F_3, G_3$ are all complex functions of the slow variables X, T . $M(X, T)$ is a real quantity to be considered as the mean flow and c.c. indicates complex conjugates. From a physical point of view this representation corresponds to a double expansion around wavenumbers k_1 and k_2 .

After substituting the expansion (2) into Eq. (1) and collecting terms for different harmonics, some lengthy but straightforward algebra leads to a set of equations for the complex envelopes and the mean flow M . At order ϵ , the equations for A or B provide the linear dispersion relation: $\omega_i = -\lambda k_i^3$ ($i=1,2$). For the second harmonics, at order ϵ^2 , the following relations hold between complex envelopes:

$$A_2 = \frac{\mu}{12\lambda k_1^2} A^2, \quad B_2 = \frac{\mu}{12\lambda k_2^2} B^2, \quad (3)$$

$$C_2 = \frac{\mu}{6\lambda k_1 k_2} AB, \quad D_2 = -\frac{\mu}{6\lambda k_1 k_2} AB^*.$$

At order ϵ^3 , the mean flow M is related to the envelopes A and B as follows:

$$M_T + \frac{\mu}{4}(|A|_X^2 + |B|_X^2) = 0. \quad (4)$$

Moreover it can be shown that D_3 and G_3 are both proportional to $1/(k_1 - k_2)^2$ and therefore the expansion is not valid when $k_1 \approx k_2$. Using Eq. (3) and neglecting terms of order higher than ϵ^3 , the equations for A and B then read

$$\epsilon^2(A_T - 3\lambda k_1^2 A_X) + i\epsilon^3 \left[3\lambda k_1 A_{XX} + \left(\mu k_1 M + \frac{\mu^2}{24\lambda} \frac{1}{k_1} |A|^2 \right) A \right] = 0, \quad (5)$$

$$\epsilon^2(B_T - 3\lambda k_2^2 B_X) + i\epsilon^3 \left[3\lambda k_2 B_{XX} + \left(\mu k_2 M + \frac{\mu^2}{24\lambda} \frac{1}{k_2} |B|^2 \right) B \right] = 0. \quad (6)$$

Considering that at ϵ^2 , $A_X \approx (1/3\lambda k_1^2) A_T$ and B_X

$\approx (1/3\lambda k_2^2) B_T$, the mean flow can be directly related to the complex envelopes A and B : integrating Eq. (4) in time we get

$$M = -\frac{\mu}{12\lambda} \left(\frac{|A|^2}{k_1^2} + \frac{|B|^2}{k_2^2} \right). \quad (7)$$

After substituting Eq. (7) in (5) and (6), we obtain

$$\epsilon^2(A_T - c_1 A_X) + i\epsilon^3[\alpha_1 A_{XX} - (\beta_1 |A|^2 + \gamma_1 |B|^2)A] = 0, \quad (8)$$

$$\epsilon^2(B_T - c_2 B_X) + i\epsilon^3[\alpha_2 B_{XX} - (\beta_2 |B|^2 + \gamma_2 |A|^2)B] = 0, \quad (9)$$

where $c_i = 3\lambda k_i^2$, $\alpha_i = 3\lambda k_i$, $\beta_i = \mu^2/24\lambda k_i$, $\gamma_1 = k_2 \mu^2/12\lambda k_1^2$, and $\gamma_2 = k_1 \mu^2/12\lambda k_2^2$. Similar equations have been recently discussed by Martel *et al.*¹⁷ In a consistent asymptotic expansion all terms must be independent of ϵ , but this is not the case for Eqs. (8) and (9). In the derivation of a single NLS equation it is always possible to select a frame of reference moving with the group velocity and therefore remove the small parameter from the equation. In the present case this operation is no longer possible because each wave envelope has its own group velocity. We account for this inconsistency in two different ways that are illustrated below.

(i) Introduce a slow time scale $\tau = \epsilon^2 t$ and let the amplitudes depend on $A = A(X, T, \tau)$ and $B = B(X, T, \tau)$, where $A = A^{(0)} + \epsilon A^{(1)}$ and $B = B^{(0)} + \epsilon B^{(1)}$. Using a standard multiple scale expansion, it is straightforward to verify that at order $O(\epsilon^2)$ the following linear equations hold:

$$A_T^{(0)} - c_1 A_X^{(0)} = 0, \quad B_T^{(0)} - c_2 B_X^{(0)} = 0. \quad (10)$$

If one introduces the two variables $\sigma = X + c_1 T$ and $\zeta = X + c_2 T$, Eq. (10) implies that $A^{(0)} = A^{(0)}(\sigma, \tau)$ and $B^{(0)} = B^{(0)}(\zeta, \tau)$. At $O(\epsilon^3)$ the following equations are obtained:

$$A_\tau^{(0)} + i\alpha_1 A_{\sigma\sigma}^{(0)} - i(\beta_1 |A^{(0)}|^2 + \gamma_1 |A^{(0)}|^2)A^{(0)} = (c_1 + c_2)A_\zeta^{(1)}, \quad (11)$$

$$B_\tau^{(0)} + i\alpha_2 B_{\zeta\zeta}^{(0)} - i(\beta_2 |B^{(0)}|^2 + \gamma_2 |A^{(0)}|^2)B^{(0)} = (c_1 + c_2)B_\sigma^{(1)}. \quad (12)$$

The solvability condition is obtained by integrating Eqs. (11) and (12) in ζ and σ over the intervals S and Z , which are the periods of $A^{(0)}$ and $B^{(0)}$, respectively (see Ref. 18 for details). The resulting asymptotic equations are

$$A_\tau^{(0)} + i\alpha_1 A_{\sigma\sigma}^{(0)} - i \left(\beta_1 |A^{(0)}|^2 + \gamma_1 \frac{1}{Z} \int_0^Z |B^{(0)}|^2 d\zeta \right) A^{(0)} = 0, \quad (13)$$

$$B_\tau^{(0)} + i\alpha_2 B_{\zeta\zeta}^{(0)} - i \left(\beta_2 |B^{(0)}|^2 + \gamma_2 \frac{1}{S} \int_0^S |A^{(0)}|^2 d\sigma \right) B^{(0)} = 0. \quad (14)$$

Note that now these equations are independent of ϵ . Plane wave solutions of system (13)–(14) are modulationally stable (see Ref. 18).

(ii) We consider again Eqs. (8) and (9) and perform the following re-scaling of time and space $\chi = \epsilon X$ and $\tau = \epsilon T$ with $A = A(\chi, \tau)$ and $B = B(\chi, \tau)$. Equations (8) and (9) re-write as

$$A_\tau - c_1 A_\chi - i\beta_1 |A|^2 A - i\gamma_1 |B|^2 A = -i\epsilon^2 \alpha_1 A_{\chi\chi}, \quad (15)$$

$$B_\tau - c_2 B_\chi - i\beta_2 |B|^2 B - i\gamma_2 |A|^2 B = -i\epsilon^2 \alpha_2 B_{\chi\chi}. \quad (16)$$

Note that if the higher order dispersive terms at the right-hand side are neglected the system is integrable: $|A|^2$ and $|B|^2$ are both conserved quantities and plane wave solutions are modulationally stable. Formally those higher order dispersive terms at the right-hand side are inconsistent with the asymptotic expansion, nevertheless the system (15) and (16) has been widely studied in different fields of physics, see Refs. 8, 17, 19–21. Here we study the influence of such term on the modulational instability of plane wave solutions. Consider a plane wave solution of (15) and (16) of the form: $A = \tilde{A} e^{-i\omega_A \tau}$ and $B = \tilde{B} e^{-i\omega_B \tau}$ with $\omega_A = -(\beta_1 |\tilde{A}|^2 + \gamma_1 |\tilde{B}|^2)$ and $\omega_B = -(\beta_2 |\tilde{B}|^2 + \gamma_2 |\tilde{A}|^2)$, i.e., the nonlinear frequency correction in the Stokes expansion. These solutions are then perturbed as follows:

$$A = \tilde{A}(1+a)e^{-i(\omega_A t + \phi_A)}, \quad B = \tilde{A}(1+b)e^{-i(\omega_B t + \phi_B)}, \quad (17)$$

where a, b, ϕ_A and ϕ_B are small perturbations in amplitude and phase. We substitute relations (17) into (15) and (16) and obtain a system for a, b, ϕ_A , and ϕ_B ; after linearization, the standard Fourier technique is used to solve it:

$$a = \tilde{a} e^{i(K\chi - \Omega\tau)}, \quad b = \tilde{b} e^{i(K\chi - \Omega\tau)},$$

$$\phi_A = \tilde{\phi}_A e^{i(K\chi - \Omega\tau)}, \quad \phi_B = \tilde{\phi}_B e^{i(K\chi - \Omega\tau)}, \quad (18)$$

where $\tilde{a}, \tilde{b}, \tilde{\alpha}$, and $\tilde{\beta}$ are constant, $\Omega = \Omega(K)$ is the dispersion relation for perturbed wavenumber K . After some algebra, the dispersion relation results in a fourth-order polynomial function in the variable Ω :

$$((\Omega + Kc_1)^2 - \epsilon^2 \alpha_1 K^2 (2\beta_1 |\tilde{A}|^2 + \epsilon^2 \alpha_1 K^2))$$

$$\times ((\Omega + Kc_2)^2 - \epsilon^2 \alpha_2 K^2 (2\beta_2 |\tilde{B}|^2 + \epsilon^2 \alpha_2 K^2))$$

$$= 4\epsilon^4 \alpha_1 \alpha_2 \gamma_1 \gamma_2 |\tilde{A}|^2 |\tilde{B}|^2 K^4. \quad (19)$$

Equation (19) provides the dispersion relation for the perturbation: complex roots, which are functions of the parameters $k_1, k_2, \tilde{A}, \tilde{B}$, and K , originate instability. Note that if we set one of the two amplitudes equal to zero (for example, $\tilde{B} = 0$), the right-hand-side term in (19) vanishes and roots can be easily found. In this case the roots are always real: this is consistent with the result that a single monochromatic wave in shallow water is stable to side-band perturbations. In the limit of $\epsilon \rightarrow 0$ with K fixed, it results that Ω has real roots; similarly, for $K \rightarrow \infty$ and fixed ϵ , plane wave solutions are stable. Moreover it is clear from Eq. (19) that if the higher order dispersive terms are neglected, plane wave solutions are stable. Equation (19) can be solved by applying the exact formula for the determination of the complex roots of a fourth-order polynomial equation. Results are graphically presented in the following way: we fix the nondimensional wavenumber $k_1 = 1$; amplitudes \tilde{A}, \tilde{B} and constants μ and λ

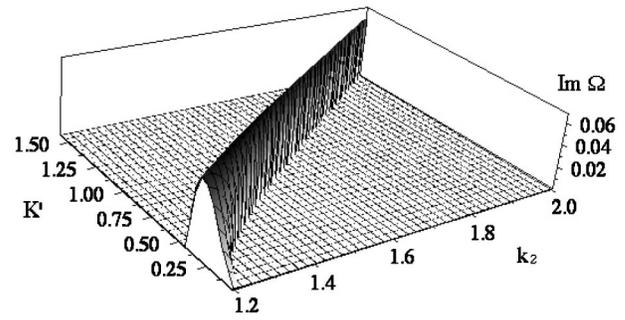


FIG. 1. Growth rate as a function of nondimensional wavenumber k_2 and perturbation $K' = K/\epsilon^2$ for $\tilde{A} = \tilde{B} = 1, k_1 = 1$ and $\epsilon = 0.8$. Axes are in nondimensional units.

are all set to 1. We then compute the largest imaginary solution of the polynomial function Ω and plot it as a surface for different values of k_2 and perturbation K . In Fig. 1 we show the instability region for $\epsilon = 0.8$. The plot clearly exhibits an unstable region with a growth rate different from zero. Note that on the horizontal axes k_2 starts from values slightly larger than 1, because, as previously stated, the derived CNLS system is not valid when $k_2 \approx k_1$. In Fig. 2 we show the same diagram as in Fig. 1 for $\epsilon = 0.4$. The smaller the value of the parameter ϵ , the smaller the size of the region of instability. However, the maximum of the growth rate, for fixed values of k_1 and k_2 , does not change substantially with ϵ . In order to check this last result we have performed direct numerical simulations of Eqs. (15) and (16). We have used a standard pseudospectral numerical method in which the linear part is solved exactly in Fourier space and the nonlinear terms are solved in physical space. Initial conditions are provided by two carrier waves at $k_1 = 1$ and $k_2 = 1.5$ of amplitude equal to one. All wavenumbers are equally perturbed with an amplitude of 0.0005 and therefore during the simulation the most unstable wavenumber is automatically selected. In Fig. 3 we show in a semi-logarithmic plot the linear stage of the evolution in time of the most unstable mode for two simulations, corresponding to $\epsilon = 0.4$ and $\epsilon = 0.8$. In the same plot we show two exponential curves

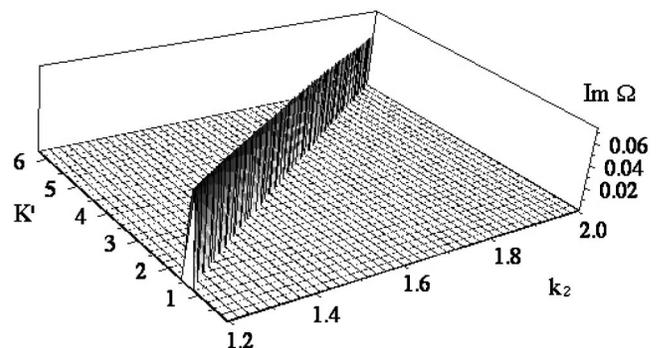


FIG. 2. Growth rate as a function of nondimensional wavenumber k_2 and perturbation $K' = K/\epsilon^2$ for $\tilde{A} = \tilde{B} = 1, k_1 = 1$, and $\epsilon = 0.4$. Axes are in nondimensional units.

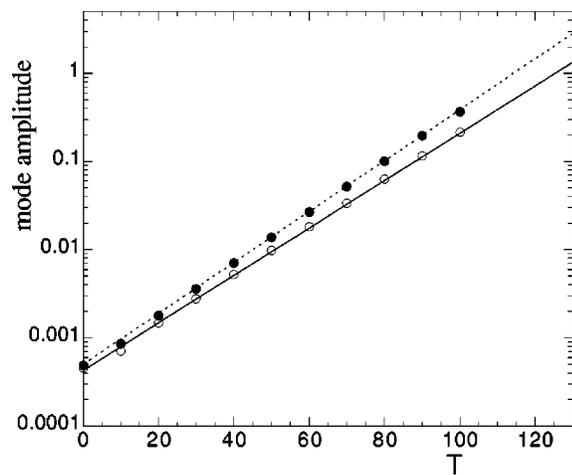


FIG. 3. Evolution in time of the most unstable wavenumber of numerical simulations with $\epsilon=0.4$ (closed circles) and $\epsilon=0.8$ (open circles). The curves correspond to the growth rate obtained from the linear stability analysis ($\exp[0.062T]$, solid line, and $\exp[0.0667T]$, dotted line).

corresponding to the growth rate predicted using the exact dispersion relation. The agreement is remarkable.

This Brief Communication represents a first investigation of the interaction of co-propagating waves in shallow water in the framework of the envelope equation approximation. The main result reported here is that a system of defocusing CNLS equation can exhibit modulational instability if dispersive terms are retained in the expansion. Delicate issues concerning proper asymptoticity have been discussed. The physical relevance of these results is the subject of ongoing research.

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