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Mathematical Modelling of a Solid–Liquid Mixture with Mass Exchange Between Constituents

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Abstract

The mechanical behavior of a mixture composed by an elastic solid and a fluid that exchange mass is investigated. Both the liquid flow and the solid deformation depend on how the solid phase has increased (diminished) its mass, i.e. on the mass conversion between constituents.

The model is developed introducing a decomposition of the solid phase deformation gradient. In particular, exploiting the criterion of maximization of the rate of dissipation, we determine an explicit evolution equation for the so–called growth tensor which involves directly the solid stress tensor.

An example of a possible choice of the constitutive functions is also presented.

Key Words: Mixtures theory, deformable porous media, growth, constitutive equations.

1 Introduction

Historically the research on flow through deformable porous solid was first developed in the branch of ground soil mechanics, soil consolidation under loading, ground water hydrology, petroleum engineering and extraction. However, the flow of liquid through deformable media is a problem that has relevance in various other contexts such as magma mechanics, paper pulp rolling, fabric dying and drying, coffee brewing, composite materials manufacturing and biomechanics. In the human body itself several subsystems can be devised that can be suitably described as deformable porous media permeated by organic liquids (articular cartilages, arteries, heart, lungs, brain and soft tissues in general). In principle, to model such bodies one should treat them as growing deformable porous solids and study the flow of a mixture of organic liquids through them.

This paper is concerned with the modelling of the mechanical behavior of a mixture made up of two constituents (a fluid and an elastic porous solid) that exchange mass, i.e. either part of the fluid is converted into solid which grows or the solid diminishes and part of its mass becomes fluids. Obviously the total mass of the body is preserved.

The modelling develops considering the body as a continuum mixture, with a focus on the solid phase, i.e. on the mechanical aspects of solid growth (resorption) process. In the one–component framework, a relevant advance in the understanding of bulk growth is due to Rodriguez *et. al.* [18], who first introduced a multiplicative decomposition of the tensor gradient of deformation into two terms, to account for growth and mechanical response separately (the reader is also referred to [10]). The same idea is developed and exploited by Rajagopal and coworkers in a number of different applications, and takes the name of theory of "natural configurations" [13]–[15]. The theory, in fact, enables to separate the solid volumetric growth (and the relative changes in stress) from its actual motion. The solid component of the mixture possesses numerous "natural configurations" (stress–free states) corresponding, in a certain sense, to the growth stages. The response of the solid is elastic from each natural configuration and the rate of dissipation determines their evolution.

In the present paper we discuss the following issues:

1. The fundamental equations, based on the balance laws and conservation principles, to describe the macroscopic behavior of a heterogeneous system in which different continua interact.
2. A rational method for developing a constitutive model for a solid–fluid mixture in which mass exchanges take place.

To determine the constitutive laws that regulate the process under investigation, we make use of the criterion of maximization of the entropy production rate (see [16]). After stating the functional form of the free energy ψ and of the entropy production rate ξ (depending on the material at hand), constitutive equations are obtained thanks a constrained maximization of ξ. In particular, we choose a quadratic form with constant coefficients for the entropy production rate.

Such an assumption is fully consistent with linear phenomenological relations that satisfy the Onsager criterion (see [11] and [12]) and does not produce field equations that do not agree with the equations of balance of mass, momentum and energy [3]. We however remark that there should be no confusion between the minimum entropy production criterion that characterizes the steady state under given constraints and for given linear phenomenological relations (with constant coefficients) satisfying the Onsager relations, and the maximum rate of entropy production criterion, that characterizes the linear phenomenological relations among all choices of constitutive equations that are related to a quadratic entropy production. To emphasize how different are the two approaches, one should note that Barbera [3] writes a a balance equation for the entropy field *and* uses the entropy inequality as a tool to analyze constitutive laws by the Prigogine principle. The maximum entropy production principle is instead a way to provide a constitutive relationship for the evolution of the natural state, as is detailed in the paper. No primitive equation exists for this purpose and therefore non contraddiction can occur.

Although no specific applications are considered here, some illustrative examples concerning particular constitutive laws are analyzed. In Section 2 we present the fundamental equations for heterogeneous media following the mixtures theory approach. We discuss the kinematics of growth (resorption) for mixtures which exchange mass (Section 2.2). Considering then the specific case of a binary mixture we apply the maximization of rate of dissipation criterion to determine the evolution of the growth tensor. Section 3 is devoted to the analysis of peculiar forms for both the Helmholtz free energy and the rate of dissipation of the system.

2 Flow through a deformable growing porous medium

The aim of this section is to develop a mathematical model for the flow of a fluid through a deformable porous medium which, due to the mass exchanges with the liquid, is subject to a growing (resorption) process.

2.1 Mixtures theory: Basic assumptions, definitions and balance equations

Modelling the behavior of complex systems in which different continua interact at microscopic level is not an easy task. There is basically one method for achieving this purpose: mixture theory.

Great strides have been made in developing a rational theoretical basis for studying the mechanics of mixtures. Starting from the pioneering works of Truesdell [19], [20] we just mention the papers by Bowen [4]–[6], Green and Naghdi [8], [9] and the volume by Rajagopal and Tao [17].

The basic premise of the theory is that the space occupied by a mixture can be considered occupied co–jointly by the various constituents of the mixture, each considered as a continuum in its own right. Thus, at each point in the space occupied by the mixture, there will be a particle belonging to each constituent.

This approach (also called *effective media* approach or *continuum mechanics* approach) assumes that the pore size is much smaller than any other characteristic length involved in the problem. In this case actual material points are hardly identifiable from a macroscopic point of view and this justifies the assumption of interpenetrating continua. In other words, mixtures theory assumes that the constituents are dense enough so that they can be homogenized as a continuum.

Each of these homogenized continua move relative to the others, due the application of forces. For each constituents one can define its motion, mass density, stress tensor, internal energy, temperature, entropy and other relevant physical quantities as one does for a single homogeneous continuum.

Let us consider a mixture of N immiscible constituents. The index α, $\alpha = 1, ..., N$, indicates the single constituent. The motion of the α^{th} constituent is described by the position occupied, at time t, by the particle labelled $\vec{X}^{\,\alpha}$

$$\vec{x} = \vec{\chi}^{\,\alpha}\left(\vec{X}^{\,\alpha},\, t\right)\,, \tag{1}$$

being $\vec{X}^{\alpha}$ the position of the particle in the reference configuration κ_o^{α} relative to the α^{th} component. The function $\vec{\chi}^{\,\alpha}$ in (1) is also called placement function[1], and represents a mapping between κ_o^{α} and the current configuration, i.e. the one that occurs at time t, denoted by κ_t^{α}. We can assume that the inverse of $\vec{\chi}^{\,\alpha}$, denoted by $\vec{\varkappa}^{\,\alpha}$, exists. Therefore

$$\vec{X}^{\,\alpha} = \vec{\varkappa}^{\,\alpha}\left(\vec{x}, t\right). \tag{2}$$

The spatial description of the velocity field is

$$\vec{v}^{\,\alpha}\left(\vec{x}, t\right) = \left.\frac{\partial \vec{\chi}^{\,\alpha}\left(\vec{X}^{\,\alpha}, t\right)}{\partial t}\right|_{\vec{X}^{\,\alpha} = \vec{\varkappa}^{\,\alpha}(\vec{x}, t)}\,, \tag{3}$$

while the material derivative following the α^{th} component of the mixture, is

$$\frac{D^{\alpha}\left(\,\cdot\,\right)}{Dt} = \frac{\partial\left(\,\cdot\,\right)}{\partial t} + \left(\vec{v}^{\,\alpha}\left(\vec{x}, t\right)\cdot\nabla\right)\left(\,\cdot\,\right), \tag{4}$$

where ∇ denotes the gradient operator with respect to the spatial position $\vec{x}$. For each constituent we introduce the deformation gradient at $\left(\vec{X}^{\alpha}, t\right)$ with respect to the κ_o^{α} configuration as the linear transformation

$$\mathbf{F}^{\alpha} = Grad^{\alpha}\vec{\chi}^{\,\alpha}\left(\vec{X}^{\,\alpha}, t\right)\,.$$

Here and in the sequel $Grad^{\alpha}$ denotes the differentiation with respect to $\vec{X}^{\,\alpha}$, i.e. the gradient operator acting with respect to $\vec{X}^{\,\alpha}$.

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[1] In abstract language $\vec{\chi}^{\,\alpha}$ is C^2 diffeomorphism.

The velocity gradient relative to the α^{th} component of the mixture is defined as

$$\mathbf{L}^{\alpha} = \nabla \vec{v}^{\,\alpha} \left(\vec{x}, t \right). \tag{5}$$

In particular, using the chain rule

$$\mathbf{L}^{\alpha} \left(\vec{x}, t \right) = \frac{D^{\alpha} \mathbf{F}^{\alpha}}{Dt} \left(\vec{x}, t \right) \, \mathbf{F}^{\alpha^{-1}} \left(\vec{x}, t \right). \tag{6}$$

The density of the α^{th} constituent is denoted by ρ^{α} (physically ρ^{α} represents the mass of the α^{th} constituent per unit volume of the mixture). The density of the mixture (sometimes called bulk density) is defined as

$$\rho = \sum_{\alpha=1}^{N} \rho^{\alpha} \left(\vec{x}, t \right). \tag{7}$$

The density for the α^{th} constituent in a homogeneous state is denoted by ρ_R^{α}. It represents the mass of the α^{th} constituent per unit volume of the α^{th} constituent. The quantity defined by

$$\phi^{\alpha} \, \left(\vec{x}, t \right) = \frac{\rho^{\alpha} \left(\vec{x}, t \right)}{\rho_R^{\alpha} \left(\vec{x}, t \right)}, \tag{8}$$

is the volume fraction of the mixture occupied by the α^{th} constituent. Because of saturation, volume ratios are subject to the following constraint[2]

$$\sum_{\alpha=1}^{N} \phi^{\alpha} \left(\vec{x}, t \right) = 1 \,. \tag{9}$$

In the following we focus only mixtures whose constituents are incompressible in their “pure state”, namely

$$\rho_R^{\alpha} = \text{const.} \;\; \alpha = 1, ..., N. \tag{10}$$

The velocity of the mixture $\vec{v}$ at $(\vec{x}, t)$ is defined by

$$\vec{v} = \frac{1}{\rho} \sum_{\alpha=1}^{N} \rho^{\alpha} \, \vec{v}^{\,\alpha}. \tag{11}$$

The diffusion velocity of the α^{th} constituent $\vec{u}^{\,\alpha}$ with respect to the mixture is defined by

$$\vec{u}^{\,\alpha} = \vec{v}^{\,\alpha} - \vec{v}. \tag{12}$$

It is useful to define the material time derivative following the mixture

$$\frac{D \left(\cdot \right)}{Dt} = \frac{\partial \left(\cdot \right)}{\partial t} + \left(\vec{v} \left(\vec{x}, t \right) \cdot \nabla \right) \left(\cdot \right). \tag{13}$$

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[2] Recall that the constituents are immiscible.

From definition (4) one has

$$\frac{D^{\alpha}\,(\,\cdot\,)}{Dt} - \frac{D\,(\,\cdot\,)}{Dt} = (\vec{u}^{\,\alpha}\cdot\nabla)\,(\,\cdot\,)\,. \tag{14}$$

According to the Eulerian formalism mass, linear momentum and energy balance equations for the α^{th} constituent of the mixture are

$$\frac{\partial\rho^{\alpha}}{\partial t} + \nabla\cdot(\rho^{\alpha}\,\vec{v}^{\,\alpha}) = \rho^{\alpha}\Gamma^{\alpha}\,, \tag{15}$$

$$\frac{\partial\rho^{\alpha}\vec{v}^{\,\alpha}}{\partial t} + \nabla\cdot(\rho^{\alpha}\,\vec{v}^{\,\alpha}\otimes\vec{v}^{\,\alpha}) = \nabla\cdot\mathbf{T}^{\alpha} + \rho^{\alpha}\vec{b} + \vec{m}^{\alpha} + \rho^{\alpha}\Gamma^{\alpha}\vec{v}^{\,\alpha}\,, \tag{16}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left[\rho^{\alpha}\left(\varepsilon^{\alpha}+\frac{|\vec{v}^{\,\alpha}|^{2}}{2}\right)\right] \quad = \quad & -\nabla\cdot\left[\rho^{\alpha}\left(\varepsilon^{\alpha}+\frac{|\vec{v}^{\,\alpha}|^{2}}{2}\right)\vec{v}^{\,\alpha}\right]- \\ & \nabla\cdot(\mathbf{T}^{\alpha}\,\vec{v}^{\,\alpha}-\vec{q}^{\alpha})+\rho^{\alpha}\vec{b}\cdot\vec{v}^{\,\alpha}+ \\ & \vec{m}^{\alpha}\cdot\vec{v}^{\,\alpha}+s^{\alpha}+\rho^{\alpha}r^{\alpha}+ \\ & \rho^{\alpha}\Gamma^{\alpha}\left(\varepsilon^{\alpha}+\frac{|\vec{v}^{\,\alpha}|^{2}}{2}\right),\end{aligned} \tag{17}$$

where:

- $\rho^{\alpha}\Gamma^{\alpha}$ is the mass supply rate for the α^{th} constituent. It accounts for possible mass conversions between constituents.
- $\mathbf{T}^{\alpha}=\mathbf{T}^{\alpha^{T}}$ is the Cauchy partial stress tensor, i.e. the stress tensor of the α^{th} constituent when the other components are co-present.
- $\rho^{\alpha}\vec{b}$ is the body force.
- $\vec{m}^{\alpha}$ is the momentum supply (also named internal interaction force) and is related to interactions between constituents that take place on their own interfaces.
- ε^{α} is the specific internal energy of the α^{th} component.
- $\vec{q}^{\alpha}$ is the partial energy flux vector associated with the α^{th} component.
- r^{α} is the external energy supply density relative to the α^{th} component.
- s^{α} is the energy supply due to energy exchanges between constituents across the interface separating them.

Taking (16) into account, equation (17) rewrites as

$$\rho^{\alpha}\frac{D^{\alpha}\varepsilon^{\alpha}}{Dt}=\mathrm{tr}\left(\mathbf{T}^{\alpha}\mathbf{L}^{\alpha}\right)+s^{\alpha}+\rho^{\alpha}r^{\alpha}-\nabla\cdot\vec{q}^{\,\alpha}. \tag{18}$$

We also remark that the mass balance equation can be written as

$$\frac{D\rho^{\alpha}}{Dt}=-\vec{u}^{\,\alpha}\cdot\nabla\rho^{\alpha}-\rho^{\alpha}\mathrm{tr}\mathbf{L}^{\alpha}+\rho^{\alpha}\Gamma^{\alpha}, \tag{19}$$

or as[3]

$$\frac{D^{\alpha}}{Dt}\left(\rho^{\alpha}J^{\alpha}\right)=\Gamma^{\alpha}\rho^{\alpha}J^{\alpha}, \tag{20}$$

where

$$J^{\alpha}=\det\mathbf{F}^{\alpha}. \tag{21}$$

In particular, if we denote by dV_o^{α} and by dV^{α} the volume occupied by a generic particle of the α^{th} constituent in the reference configuration κ_o^{α} and in the current configuration κ_t^{α}, we have

$$dV_o^{\alpha}=\frac{dV^{\alpha}}{J^{\alpha}}. \tag{22}$$

Thus, exploiting assumption (10), we can introduce the quantity

$$\mathcal{M}^{\alpha}=\frac{\rho_R^{\alpha}\phi^{\alpha}dV^{\alpha}}{\rho_R^{\alpha}\phi_o^{\alpha}dV_o^{\alpha}}=\frac{\phi^{\alpha}}{\phi_o^{\alpha}}J^{\alpha}, \tag{23}$$

which measures the volumetric growth ($\mathcal{M}^{\alpha}>1$) or resorption ($\mathcal{M}^{\alpha}<1$) of the α^{th} constituent with respect to the configuration κ_o^{α} (ϕ_o^{α} is the volume fraction of the α^{th} constituent in the configuration κ_o^{α}).

In particular, from (20) we get

$$\frac{1}{\mathcal{M}^{\alpha}}\frac{D^{\alpha}\mathcal{M}^{\alpha}}{Dt}=\Gamma^{\alpha}, \tag{24}$$

whose solution, considering that $\mathcal{M}^{\alpha}\left(\vec{X}^{\alpha},0\right)=1$, yields

$$\mathcal{M}^{\alpha}\left(\vec{X}^{\alpha},t\right)=\exp\left\{\int_0^t\Gamma^{\alpha}d\tau\right\}. \tag{25}$$

So, if the constitutive equation for Γ^{α} and the evolution in time of the related fields is known, (25) gives an explicit expression for the volumetric growth of

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[3] Recall that $\frac{\partial}{\partial\mathbf{A}}\left(\det\mathbf{A}\right)=\left(\det\mathbf{A}\right)\mathbf{A}^{-T}$ and that $\frac{D}{Dt}\left(\det\mathbf{A}\right)=\left(\det\mathbf{A}\right)\mathrm{tr}\left(\frac{D\mathbf{A}}{Dt}\mathbf{A}^{-1}\right)$.

the α^{th} constituent. From equations (15)–(17), assuming

$$\sum_{\alpha=1}^{N} \rho^\alpha \Gamma^\alpha = 0\,, \tag{26}$$

$$\sum_{\alpha=1}^{N} (\rho^\alpha \Gamma^\alpha \vec{v}^{\,\alpha} + \vec{m}^\alpha) = 0, \tag{27}$$

$$\sum_{\alpha=1}^{N} \left[s^\alpha + \rho^\alpha \Gamma^\alpha \left(\varepsilon^\alpha + \frac{|\vec{v}^{\,\alpha}|^2}{2} \right) + \vec{m}^{\,\alpha} \cdot \vec{v}^{\,\alpha} \right] = 0, \tag{28}$$

the balance equations of the mixture as a whole can be easily obtained:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \vec{v}\,) = 0\,, \tag{29}$$

$$\rho\, \frac{D\vec{v}}{Dt} = \nabla \cdot \mathbf{T} + \rho \vec{b}, \tag{30}$$

$$\rho \frac{D\varepsilon}{Dt} = \sum_{\alpha=1}^{N} \Big[\mathrm{tr}\,(\mathbf{T}^\alpha \mathbf{L}^\alpha) - \nabla \cdot (\vec{q}^{\,\alpha} + \rho^\alpha \varepsilon^\alpha \vec{u}^{\,\alpha}\,) - \vec{v}^{\,\alpha} \cdot \vec{m}^{\,\alpha} - \rho^\alpha \Gamma^\alpha \frac{|\vec{v}^{\,\alpha}|^2}{2} \Big] + \rho r, \tag{31}$$

where:

- $\mathbf{T}$ is the total stress tensor of the mixture

$$\mathbf{T} = \sum_{\alpha=1}^{N} (\mathbf{T}^\alpha - \rho^\alpha \vec{u}^{\,\alpha} \otimes \vec{u}^{\,\alpha})\ . \tag{32}$$

- ε is the internal energy density of the mixture

$$\varepsilon = \frac{1}{\rho} \sum_{\alpha=1}^{N} \rho^\alpha \varepsilon^\alpha\ . \tag{33}$$

- r is the specific external supply energy

$$r = \frac{1}{\rho} \sum_{\alpha=1}^{N} \rho^\alpha r^\alpha. \tag{34}$$

We assume that the global form of the entropy inequality for the mixture as a whole (Clausius–Duhem inequality for the mixture) is given by

$$\frac{\partial}{\partial t}\left(\int\limits_{\mathcal{U}}\sum_{\alpha=1}^{N}\rho^\alpha\eta^\alpha d^3\vec{x}\right)+\int\limits_{\partial\mathcal{U}}\left(\sum_{\alpha=1}^{N}\rho^\alpha\eta^\alpha\,\vec{v}^{\,\alpha}\cdot\vec{n}\,ds\right)\quad\geq$$
$$\int\limits_{\partial\mathcal{U}}\left(\sum_{\alpha=1}^{N}-\frac{\vec{h}^{\,\alpha}}{\theta}\cdot\vec{n}\,ds\right)+\int\limits_{\mathcal{U}}\left(\sum_{\alpha=1}^{N}\frac{\rho^\alpha r^\alpha}{\theta}d^3\vec{x}\right), \tag{35}$$

where:

- η^α is the specific entropy of the α^{th} component.
- $\vec{n}$ is the outward normal to $\partial\mathcal{U}$.
- $\vec{h}^{\,\alpha}$ is the entropy flux (not yet related to $\vec{q}^{\alpha}$).
- θ is the absolute temperature which is assumed to be the same for all the constituents.

If we now define the specific entropy for the mixture as

$$\eta=\frac{1}{\rho}\sum_{\alpha=1}^{N}\rho^\alpha\eta^\alpha, \tag{36}$$

inequality (35) can be rewritten as

$$\rho\frac{D\eta}{Dt}\geq\sigma_e, \tag{37}$$

where

$$\sigma_e=-\nabla\cdot\sum_{\alpha=1}^{N}\left(\frac{\vec{h}^{\,\alpha}+\rho^\alpha\eta^\alpha\theta\vec{u}^{\,\alpha}}{\theta}\right)+\frac{\rho r}{\theta}. \tag{38}$$

For what the physics is concerned, σ_e is usually interpreted as the entropy production rate due to the interactions with surroundings. From (37) we deduce that there must be a non–negative σ_i such that

$$\rho\frac{D\eta}{Dt}=\sigma_e+\sigma_i. \tag{39}$$

Within the context of classical thermodynamics, $\sigma_i\geq 0$ represents the so–called uncompensated heat and expresses the internal entropy production rate per unit volume. Thus equation (39) takes the form

$$\rho\theta\frac{D\eta}{Dt}=-\theta\nabla\cdot\left(\sum_{\alpha=1}^{N}\frac{\vec{q}^{\alpha}+\rho^\alpha\eta^\alpha\theta\,\vec{u}^{\,\alpha}}{\theta}\right)+\rho r+\theta\sigma_i. \tag{40}$$

We now introduce the rate of entropy production for the mixture

$$\xi = \sigma_i \theta \geq 0, \tag{41}$$

and obtain

$$\begin{aligned}\xi &= \rho\theta\frac{D\eta}{Dt} + \nabla\cdot\sum_{\alpha=1}^{N}\left(\vec{h}^{\,\alpha} + \rho^\alpha\eta^\alpha\theta\,\vec{u}^{\,\alpha}\right) - \rho r + \\ &\quad \theta\nabla\left(\frac{1}{\theta}\right)\cdot\sum_{\alpha=1}^{N}\left(\vec{h}^{\,\alpha} + \rho^\alpha\eta^\alpha\theta\,\vec{u}^{\,\alpha}\right).\end{aligned} \tag{42}$$

We define the specific Helmholtz free energy for the α^{th} constituent

$$\psi^\alpha = \varepsilon^\alpha - \theta\eta^\alpha\,, \tag{43}$$

and for the mixture as a whole

$$\psi = \frac{1}{\rho}\sum_{\alpha=1}^{N}\rho^\alpha\psi^\alpha, \tag{44}$$

and write

$$\psi = \varepsilon - \theta\eta, \tag{45}$$

$$\frac{D\psi}{Dt} = \frac{D\varepsilon}{Dt} - \theta\frac{D\eta}{Dt} - \eta\frac{D\theta}{Dt}. \tag{46}$$

Considering isothermal conditions, i.e. θ is constant in space and time, and recalling the energy balance (31), the rate of entropy production ξ takes the form

$$\begin{aligned}\xi &= \rho\frac{D\varepsilon}{Dt} + \nabla\cdot\sum_{\alpha=1}^{N}\left(\vec{h}^{\,\alpha} + \rho^\alpha\varepsilon^\alpha\vec{u}^{\,\alpha}\right) - \nabla\cdot\sum_{\alpha=1}^{N}\rho^\alpha\psi^\alpha\vec{u}^{\,\alpha} - \\ &\quad \rho r - \rho\frac{D\psi}{Dt} \\ &= \sum_{\alpha=1}^{N}\left[\mathrm{tr}\left(\mathbf{T}^\alpha\mathbf{L}^\alpha\right) - \vec{m}^{\,\alpha}\cdot\vec{v}^{\,\alpha}\right] - \nabla\cdot\sum_{\alpha=1}^{N}\left(\rho^\alpha\psi^\alpha\vec{u}^{\,\alpha} + \vec{q}^{\,\alpha} - \vec{h}^{\,\alpha}\right) - \\ &\quad \sum_{\alpha=1}^{N}\frac{\rho^\alpha\Gamma^\alpha\left|\vec{v}^{\,\alpha}\right|^2}{2} - \rho\frac{D\psi}{Dt}.\end{aligned} \tag{47}$$

The above equation is a possible starting point for the determination of the constitutive equations. We will use (47) to place some restrictions on the structure of $\mathbf{T}^\alpha$ and $\vec{m}^{\,\alpha}$.

2.2 Kinematics of growth

As mentioned in the Introduction, we are interested in modelling mixtures in which mass exchanges among components take place, giving rise to volumetric growth (resorption).

Let us consider the motion of a generic particle of the α^{th} constituent from its reference configuration κ_o^α, which we assume to be stress free. Let us now imagine to cut such a particle out of the mixture and relieve its state of stress while keeping its mass constant. The particle will reach a state, called natural state, that is in general different from κ_o^α and also from κ_t^α. We define the natural configuration of the α^{th} constituent at time t, and we denote it by κ_p^α, the collection of all particles natural states.

We introduce the tensor $\mathbf{F}_p^\alpha$ for measuring the deformation from the natural configuration κ_p^α to κ_t^α. The path from κ_o^α to κ_p^α can be seen as the path of unconstrained growth (resorption) and will be described by the so–called growth tensor $\mathbf{G}^\alpha$. Hence the following decomposition holds[4]

$$\mathbf{F}_p^\alpha = \mathbf{F}^\alpha {\mathbf{G}^\alpha}^{-1} . \tag{48}$$

Let us note that, since mass is preserved along the path from κ_p^α to κ_t^α, the tensor $\mathbf{F}_p^\alpha$ is not directly related to growth (resorption) but it is connected with the stress response of the material. On the other hand, since from κ_o^α to κ_p^α mass is not preserved, $\mathbf{G}^\alpha$ accounts for the volumetric growth[5]. In analogy with formula (6) we define

$$\mathbf{L}_G = \frac{D^\alpha \mathbf{G}^\alpha}{Dt} {\mathbf{G}^\alpha}^{-1}, \tag{49}$$

$$\mathbf{L}_p = \frac{D^\alpha \mathbf{F}_p^\alpha}{Dt} {\mathbf{F}_p^\alpha}^{-1}. \tag{50}$$

We remark however that (49) and (50) are nothing but definitions, while (6) is a consequence of definition (5). Defining

$$J_p^\alpha = \det\left(\mathbf{F}_p^\alpha\right), \tag{51}$$

$$J_G^\alpha = \det\left(\mathbf{G}^\alpha\right), \tag{52}$$

mass conservation equation (20) acquires the form

$$\rho^\alpha J_p^\alpha \frac{D^\alpha J_G^\alpha}{Dt} + J_G^\alpha \frac{D^\alpha \left(\rho^\alpha J_p^\alpha\right)}{Dt} = \rho^\alpha\, \Gamma^\alpha J_p^\alpha\, J_G^\alpha . \tag{53}$$

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[4] As $\mathbf{F}^\alpha$ is invertible, from (48) it follows that $\mathbf{F}_p^\alpha$ and $\mathbf{G}^\alpha$ are invertible too.

[5] We remark that $\mathbf{G}^\alpha$ need not to be differential of any particular functions, as $\mathbf{F}^\alpha$ does. In other words it does not exist, in general, a global mapping $\vec{\zeta}^{\,\alpha}$ from κ_o^α to κ_p^α such that

$$\mathbf{G}^\alpha = Grad^\alpha \vec{\zeta}^{\,\alpha}\left(\vec{X}^{\,\alpha}, t\right) .$$

We now assume that [1]

$$\frac{D^\alpha \left(\rho^\alpha J_p^\alpha\right)}{Dt} = 0, \tag{54}$$

which, in the framework of classical theory, represents mass conservation in going from κ_p^α to κ_t^α. Hence (53) reduces to

$$\frac{D^\alpha J_G^\alpha}{Dt} = \Gamma^\alpha J_G^\alpha, \tag{55}$$

which, recalling (49) and footnote 3, yields

$$\mathrm{tr}\mathbf{L}_G^\alpha = \Gamma^\alpha. \tag{56}$$

The tensor $\mathbf{G}^\alpha$ is thus sufficient to tell whether a certain particle is growing or being resorbed. Indeed, assuming $\mathbf{G}^\alpha\left(\vec{X}^\alpha, 0\right) = \mathbf{I}$, from (24) we have

$$\mathcal{M}^\alpha = J_G^\alpha, \tag{57}$$

so

$$\begin{aligned} J_G^\alpha < 1 \qquad &\Rightarrow \qquad \text{resorption}, \\ J_G^\alpha > 1 \qquad &\Rightarrow \qquad \text{growth}. \end{aligned} \tag{58}$$

Next, recalling (23), (48) and (57) entail

$$\frac{\phi^\alpha}{\phi_o^\alpha} = \frac{1}{J_p^\alpha}. \tag{59}$$

This means that the volume fraction of the of the α^{th} constituent (and consequently its density in the mixture) is expressed in terms of $\mathbf{F}_p^\alpha$. Moreover, defining

$$\rho_o^\alpha = \rho_R^\alpha \, \phi_o^\alpha, \tag{60}$$

(recall that ϕ_o^α, volume fraction of the α^{th} constituent at the initial time, is assumed to be known) we have

$$\rho^\alpha = \frac{\rho_o^\alpha}{J_p^\alpha}. \tag{61}$$

In other words, assumption (54) entails both (56) and (61), i.e. the α^{th} constituent mass supply rate and density are completely determined by $\mathbf{G}^\alpha$ and $\mathbf{F}_p^\alpha$, respectively.

2.3 Mixture of a deformable solid and a fluid that exchange mass: thermodynamical restrictions on the constitutive equations

We now consider a binary mixture of a solid and a fluid, so $\alpha = s, f$. As mentioned we consider isothermal conditions. In order to have a lighter notation here and in the sequel tensors $\mathbf{G}^s$, $\mathbf{F}_p^s$ and $\mathbf{F}^s$ will be denoted by $\mathbf{G}$, $\mathbf{F}_p$ and $\mathbf{F}$, respectively. Conditions (26) and (27) imply

$$\Gamma^f = -\frac{\rho^s}{\rho^f}\Gamma^s, \tag{62}$$

and

$$\begin{aligned} \vec{m}^f &= -\vec{m}^s - \sum_{\alpha=s,f} \rho^\alpha \Gamma^\alpha \vec{v}^\alpha \\ &= -\vec{m}^s - \rho^s\,\Gamma^s\,\vec{v}_{s,\,f}\,, \end{aligned} \tag{63}$$

where $\vec{v}_{s,\,f}$ is the velocity of the solid with respect to the fluid, i.e.

$$\vec{v}_{s,\,f} = \vec{v}^{\,s} - \vec{v}^{\,f}\,. \tag{64}$$

Further, because of (8), (9) and (10), we have

$$\rho^f = \gamma(\rho_R^s - \rho^s), \tag{65}$$

where

$$\gamma = \frac{\rho_R^f}{\rho_R^s}\,. \tag{66}$$

So, starting form equation (47), we look for both thermodynamical restrictions on $\vec{m}^s$, $\mathbf{T}^\alpha$, $\alpha = s, f$ and an evolution equation for $\mathbf{G}$.

We suppose that

$$\psi = \hat{\psi}\left(\mathbf{F}_p, \mathbf{G}, \rho_o^s, \gamma\right), \tag{67}$$

and that

$$\xi = \hat{\xi}\left(\mathbf{L}_G, \mathbf{F}_p, \mathbf{G}, \vec{v}_{s,\,f}, \rho_o^s, \gamma\right), \tag{68}$$

are given functions. We remark that, because of (65), ρ^f does not appear as it depends on ρ^s. The latter is then determined in terms of $\mathbf{F}_p$ and ρ_o^s (see equation (61)). Further, because of (56), Γ^s is not an independent variable.

It is customary to write the inequality (47) in a form such that the thermodynamical restrictions on constitutive equations can be easily obtained. To do that one has to evaluate $\dfrac{D\hat{\psi}}{Dt}$. According to (67) and (14) we obtain[6]

$$\begin{aligned} \frac{D\hat{\psi}}{Dt} &= \frac{D^s\psi}{Dt} - \vec{u}^{\,s}\cdot\nabla\psi \\ &= \frac{\partial\hat{\psi}}{\partial\mathbf{F}_p}\cdot\frac{D^s\mathbf{F}_p}{Dt} + \frac{\partial\hat{\psi}}{\partial\mathbf{G}}\cdot\frac{D^s\mathbf{G}}{Dt} \end{aligned}$$

The explicit evaluation of $\dfrac{\partial\hat{\psi}}{\partial\mathbf{F}_p}\cdot\dfrac{D^s\mathbf{F}_p}{Dt}$ is laborious. We have

$$\frac{\partial\hat{\psi}}{\partial\mathbf{F}_p}\cdot\frac{D^s\mathbf{F}_p}{Dt} = \mathrm{tr}\left(\frac{\partial\hat{\psi}}{\partial\mathbf{F}_p^T}\frac{D^s\mathbf{F}_p}{Dt}\right). \tag{69}$$

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[6] Recall that $\mathbf{A}\cdot\mathbf{B}=\mathrm{tr}\left(\mathbf{A}^T\mathbf{B}\right)=\mathrm{tr}\left(\mathbf{A}\mathbf{B}^T\right)$ since $\mathrm{tr}\left(\mathbf{C}^T\right) = \mathrm{tr}\left(\mathbf{C}\right)$.

Now recalling (48) and (6) we get

$$\begin{aligned}\frac{D^s\mathbf{F}_p}{Dt} &= \frac{D^s\mathbf{F}}{Dt}\mathbf{G}^{-1} + \mathbf{F}\frac{D^s\mathbf{G}^{-1}}{Dt} \\ &= \mathbf{L}^s\mathbf{F}\mathbf{G}^{-1} - \mathbf{F}\mathbf{G}^{-1}\frac{D^s\mathbf{G}}{Dt}\mathbf{G}^{-1} \\ &= \mathbf{L}^s\mathbf{F}_p - \mathbf{F}_p\mathbf{L}_G, \qquad (70)\end{aligned}$$

where definition (49) has been exploited. Substitution of (70) into (69) yields

$$\mathrm{tr}\left(\frac{\partial\hat{\psi}}{\partial\mathbf{F}_p^T}\frac{D^s\mathbf{F}_p}{Dt}\right) = \mathrm{tr}\left(\mathbf{F}_p\frac{\partial\hat{\psi}}{\partial\mathbf{F}_p^T}\mathbf{L}^s\right) - \mathrm{tr}\left(\mathbf{F}_p^T\frac{\partial\hat{\psi}}{\partial\mathbf{F}_p}\mathbf{L}_G^T\right). \qquad (71)$$

Operating in a similar way, the following expression for $\frac{\partial\hat{\psi}}{\partial\mathbf{G}}\cdot\frac{D^s\mathbf{G}}{Dt}$ is derived

$$\begin{aligned}\frac{\partial\hat{\psi}}{\partial\mathbf{G}}\cdot\frac{D^s\mathbf{G}}{Dt} &= \mathrm{tr}\left(\frac{\partial\hat{\psi}}{\partial\mathbf{G}}\frac{D^s\mathbf{G}^T}{Dt}\right) \\ &= \mathrm{tr}\left(\frac{\partial\hat{\psi}}{\partial\mathbf{G}}\mathbf{G}^T\mathbf{L}_G^T\right). \qquad (72)\end{aligned}$$

Now, going back to (47) and replacing $\frac{D\psi}{Dt}$ with its expression, we get

$$\begin{aligned}\xi &= -\nabla\cdot\sum_{\alpha=s,f}\left(\rho^\alpha\psi^\alpha\vec{u}^{\,\alpha} + \vec{q}^{\,\alpha} - \vec{h}^{\,\alpha}\right) - \\ &\sum_{\alpha=s,f}\vec{v}^{\,\alpha}\cdot\left(\vec{m}^{\,\alpha} + \frac{\rho^\alpha\Gamma^\alpha}{2}\vec{v}^{\,\alpha}\right) + \rho\,\vec{u}^{\,s}\cdot\nabla\hat{\psi} + \\ &\mathrm{tr}\left[\left(\mathbf{T}^s - \rho\mathbf{F}_p\frac{\partial\hat{\psi}}{\partial\mathbf{F}_p^T}\right)\mathbf{L}^s + \mathbf{T}^f\,\mathbf{L}^f\right] + \\ &\mathrm{tr}\left[\rho\left(\mathbf{F}_p^T\frac{\partial\hat{\psi}}{\partial\mathbf{F}_p} - \frac{\partial\hat{\psi}}{\partial\mathbf{G}}\mathbf{G}^T\right)\mathbf{L}_G^T\right] \geq 0. \qquad (73)\end{aligned}$$

Now taking (56), (62) and (63) into account, (73) assumes the form

$$\begin{aligned}\xi \;=\;& -\nabla\cdot\sum_{\alpha=s,f}\left(\rho^{\alpha}\psi^{\alpha}\vec{u}^{\,\alpha}+\vec{q}^{\,\alpha}-\vec{h}^{\,\alpha}\right)+\\ & \operatorname{tr}\left[\left(\mathbf{T}^{s}-\rho\mathbf{F}_{p}\frac{\partial\hat{\psi}}{\partial\mathbf{F}_{p}^{T}}\right)\mathbf{L}^{s}+\mathbf{T}^{f}\,\mathbf{L}^{f}\right]+\\ & \operatorname{tr}\left[\rho\left(\mathbf{F}_{p}^{T}\frac{\partial\hat{\psi}}{\partial\mathbf{F}_{p}}-\frac{\partial\hat{\psi}}{\partial\mathbf{G}}\mathbf{G}^{T}\right)\mathbf{L}_{G}^{T}\right]-\\ & \left[\vec{m}^{\,s}+\frac{\rho^{s}\vec{v}_{s,\,f}}{2}\operatorname{tr}\mathbf{L}_{G}-\gamma\left(\rho_{R}^{s}-\rho^{s}\right)\nabla\hat{\psi}\right]\cdot\vec{v}_{s,\,f}, \qquad (74)\end{aligned}$$

where we have exploited the relation $\rho\vec{u}^{s}=\rho^{f}\vec{v}_{s,\,f}=\gamma\left(\rho_{R}^{s}-\rho^{s}\right)\vec{v}_{s,\,f}$.

The assumption that the mixture constituents are incompressible in their "pure state" implies (9). Such a constraint leads to a Lagrange multiplier into the field equations, and the constraint does no work. It readily follows, adding the mass balance equations (15), that

$$\nabla\cdot\left(\phi^{s}\vec{v}^{\,s}+\phi^{f}\vec{v}^{\,f}\right)=\phi^{s}\Gamma^{s}+\phi^{f}\,\Gamma^{f}, \qquad (75)$$

i.e.

$$\phi^{s}\operatorname{tr}\mathbf{L}^{s}+\phi^{f}\operatorname{tr}\mathbf{L}^{f}+\phi^{s}\left(\frac{1-\gamma}{\gamma}\right)\operatorname{tr}\mathbf{L}_{G}+\nabla\phi^{s}\cdot\vec{v}_{s,\,f}=0\,, \qquad (76)$$

with γ given by (66).

Since equation (74) is expected to hold under the above constraint, a Lagrange multiplier λ is introduced to take account for the constraint, thus leading to

$$\begin{aligned}\xi \;=\;& \sum_{\alpha=s,f}\nabla\cdot\left[\vec{h}^{\,\alpha}-\left(\vec{q}^{\,\alpha}+\rho^{\alpha}\psi^{\alpha}\vec{u}^{\,\alpha}\right)\right]+\\ & \operatorname{tr}\left[\left(\mathbf{T}^{s}-\rho\mathbf{F}_{p}\frac{\partial\hat{\psi}}{\partial\mathbf{F}_{p}^{T}}+\lambda\phi^{s}\mathbf{I}\right)\mathbf{L}^{s}\right]+\\ & \operatorname{tr}\left[\left(\mathbf{T}^{f}+\lambda\phi^{f}\mathbf{I}\right)\mathbf{L}^{f}\right]+\\ & \operatorname{tr}\left\{\left[\rho\left(\mathbf{F}_{p}^{T}\frac{\partial\hat{\psi}}{\partial\mathbf{F}_{p}}-\frac{\partial\hat{\psi}}{\partial\mathbf{G}}\mathbf{G}^{T}\right)+\rho^{s}\lambda\frac{1-\gamma}{\rho_{R}^{f}}\mathbf{I}\right]\mathbf{L}_{G}^{T}\right\}-\\ & \left[\vec{m}^{\,s}+\frac{\rho^{s}\vec{v}_{s,\,f}}{2}\operatorname{tr}\mathbf{L}_{G}-\gamma\left(\rho_{R}^{s}-\rho^{s}\right)\nabla\hat{\psi}-\lambda\nabla\phi^{s}\right]\cdot\vec{v}_{s,\,f}\,. \qquad (77)\end{aligned}$$

Note that the above equation suggests that the constitutive assumptions for $\mathbf{T}^{f}$, $\mathbf{T}^{s}$ and $\vec{m}^{s}$ must be consistent with the entropy inequality. In particular, it can be shown that λ is related to the isotropic part of $\mathbf{T}$ and therefore physically interpreted as an hydrostatic pressure acting on the mixture.

Following standard practice (see, for instance, [4] and [17]), we make the following assumption

$$\vec{h}^{\,\alpha} = \vec{q}^{\,\alpha} + \rho^{\alpha}\psi^{\alpha}\vec{u}^{\,\alpha}, \tag{78}$$

which, in the context of mixture theory, means that the entropy flux depends also on the diffusive velocity, i.e. on the relative “mixing” of the components.

Recalling now that $\xi = \hat{\xi}\left(\mathbf{L}_G, \mathbf{F}_p, \mathbf{G}, \vec{v}_{s,\,f}, \rho_o^s, \gamma\right)$, the second and the third term on the r.h.s. of (77) are the only ones that depend on $\mathbf{L}^s$ and $\mathbf{L}^f$. Hence

$$\begin{aligned} \mathbf{T}^s &= -\lambda\phi^s\mathbf{I} + \rho\mathbf{F}_p\frac{\partial\hat{\psi}}{\partial\mathbf{F}_p^T}, && (79) \\ \mathbf{T}^f &= -\lambda\phi^f\mathbf{I}. && (80) \end{aligned}$$

We now assume that

$$\hat{\xi}\left(\mathbf{L}_G, \mathbf{F}_p, \mathbf{G}, \vec{v}_{s,\,f}, \rho_o^s, \gamma\right) = \hat{\xi}_G\left(\mathbf{L}_G, \mathbf{F}_p, \mathbf{G}, \rho_o^s, \gamma\right) + \vec{v}_{s,\,f}\cdot\mathbf{H}\,\vec{v}_{s,\,f}, \tag{81}$$

where:

- $\hat{\xi}_G$ represents the rate of entropy production due to the mass conversion between solid and fluid (i.e. due to the solid growth–resorption process).
- $\vec{v}_{s,\,f}\cdot\mathbf{H}\,\vec{v}_{s,\,f}$ represents the rate at which entropy is produced due to the solid–fluid interactions. In particular $\mathbf{H}$ is a second order symmetric positive definite tensor. In general $\mathbf{H}$ depends on $\mathbf{L}_G$, $\mathbf{F}_p$, $\mathbf{G}$, ρ_o^s and γ. Moreover, for any $\mathbf{Q}\in$Orth$^+$, i.e. for any rotation, $\mathbf{H}$ must satisfy the following transformation rule

  $$\mathbf{H}^* = \mathbf{Q}\,\mathbf{H}\,\mathbf{Q}^T. \tag{82}$$

According to the above assumptions, (77) reduces to

$$\begin{aligned} \hat{\xi}_G\left(\mathbf{L}_G, \mathbf{F}_p, \mathbf{G}, \rho_o^s, \gamma\right) \;=\; & \operatorname{tr}\left\{\left[\rho\left(\mathbf{F}_p^T\frac{\partial\hat{\psi}}{\partial\mathbf{F}_p} - \frac{\partial\hat{\psi}}{\partial\mathbf{G}}\mathbf{G}^T\right) + \right.\right. \\ & \left.\left.\rho^s\lambda\frac{1-\gamma}{\rho_R^f}\mathbf{I}\right]\mathbf{L}_G^T\right\} - \left[\vec{m}^{\,s} + \frac{\rho^s\vec{v}_{s,\,f}}{2}\operatorname{tr}\mathbf{L}_G - \right. \\ & \left.\gamma\left(\rho_R^s - \rho^s\right)\nabla\hat{\psi} - \lambda\nabla\phi^s - \mathbf{H}\,\vec{v}_{s,\,f}\right]\cdot\vec{v}_{s,\,f}\,. \end{aligned} \tag{83}$$

As one can see, in the l.h.s. of (83) there is no dependence on $\vec{v}_{s,\,f}$, so

$$\vec{m}^{\,s} = \left(\mathbf{H} - \frac{\rho^s}{2}\left(\operatorname{tr}\mathbf{L}_G\right)\mathbf{I}\right)\vec{v}_{s,\,f} + \gamma\left(\rho_R^s - \rho^s\right)\nabla\hat{\psi} + \lambda\nabla\phi^s, \tag{84}$$

and equation (83) becomes

$$\hat{\xi}_G\left(\mathbf{L}_G, \mathbf{F}_p, \mathbf{G}, \rho_o^s, \gamma\right) = \operatorname{tr}\left[\rho\left(\mathbf{F}_p^T\frac{\partial\hat{\psi}}{\partial\mathbf{F}_p} - \frac{\partial\hat{\psi}}{\partial\mathbf{G}}\mathbf{G}^T\right) + \rho^s\lambda\frac{1-\gamma}{\rho_R^f}\mathbf{I}\right]\mathbf{L}_G^T, \tag{85}$$

where $\hat{\xi}_G$ is a positive scalar-valued function to be constitutively specified.

In order to get the evolution of tensor $\mathbf{G}$, we make use of the criterion of the maximization of the rate of dissipation $\hat{\xi}$. Recalling that $\hat{\xi}_G$ and $\hat{\psi}$ are given functions (see (67) and (68)), we look for $\mathbf{L}_G$ that maximize $\hat{\xi}_G$ under the constraint (85), keeping $\mathbf{F}_p$, $\mathbf{G}$, ρ_o^s and γ as fixed. We get

$$\frac{\partial \hat{\xi}_G}{\partial \mathbf{L}_G} + \mu \left[\frac{\partial \hat{\xi}_G}{\partial \mathbf{L}_G} - \rho \left(\mathbf{F}_p^T \frac{\partial \hat{\psi}}{\partial \mathbf{F}_p} - \frac{\partial \hat{\psi}}{\partial \mathbf{G}} \mathbf{G}^T \right) - \rho^s \lambda \frac{1-\gamma}{\rho_R^f} \mathbf{I} \right] = 0, \tag{86}$$

where μ is a Lagrange multiplier which has to be determined by the satisfaction of (85). Solving (86) we obtain

$$\mathbf{L}_G = \mathbf{L}_G \left(\mathbf{F}_p, \mathbf{G}, \rho_o^s, \gamma \right). \tag{87}$$

Upon substituting into (83) we recover μ

$$\mu = \mu \left(\mathbf{F}_p, \mathbf{G}, \rho_o^s, \gamma, \lambda \right). \tag{88}$$

We remark that when we select $\hat{\xi}_G$ and $\hat{\psi}$ we should verify that they guarantee the existence of a unique μ once we substitute (87) into (85).

Summarizing, the unknowns to be determined are

$$\mathbf{G}, \ \ \vec{v}_s, \ \ \vec{v}_f \ \ \text{and} \ \ \lambda,$$

i.e. 15 scalar unknowns plus the Lagrange multiplier λ. Actually, the real unknowns are

$$\mathbf{G}, \ \ \vec{\chi}^s, \ \ \vec{\chi}^f \ \ \text{and} \ \ \lambda,$$

since

$$\mathbf{F}_p = \left(Grad^s \vec{\chi}^s \right) \mathbf{G}^{-1}, \tag{89}$$

and

$$\vec{v}_s = \frac{\partial \vec{\chi}^s}{\partial t}, \ \ \vec{v}_f = \frac{\partial \vec{\chi}^f}{\partial t}. \tag{90}$$

In order to determine such unknowns we need to solve the following 15 scalar differential equations

$$\frac{D^s \mathbf{G}}{Dt} = \mathbf{L}_G \left(\mathbf{F}_p, \mathbf{G}, \vec{v}_{s,\,f}, \rho_o^s, \gamma \right) \mathbf{G}, \tag{91}$$

$$\frac{\partial \rho^s \vec{v}^{\,s}}{\partial t} + \nabla \cdot \left(\rho^s \, \vec{v}^{\,s} \otimes \vec{v}^{\,s} \right) = \nabla \cdot \mathbf{T}^s + \rho^s \vec{g} + \vec{m}^s + \rho^s \Gamma^s \vec{v}^{\,s}, \tag{92}$$

$$\frac{\partial \rho^f \vec{v}^{\,f}}{\partial t} + \nabla \cdot \left(\rho^f \, \vec{v}^{\,f} \otimes \vec{v}^{\,f} \right) = \nabla \cdot \mathbf{T}^f + \rho^f \vec{g} - \vec{m}^s - \rho^s \Gamma^s \vec{v}_{s,\,f}\,, \tag{93}$$

plus the constraint

$$\phi^s \, \mathrm{tr} \, \mathbf{L}^s + \phi^f \, \mathrm{tr} \, \mathbf{L}^f + \phi^s \left(\frac{1-\gamma}{\gamma} \right) \mathrm{tr} \, \mathbf{L}_G + \nabla \phi^s \cdot \vec{v}_{s,\,f} = 0\,. \tag{94}$$

System (91)–(94) has to be solved, bearing in mind relations (63), (79), (80), (84), (89), (90), $\Gamma^s = \mathrm{tr}(\mathbf{L}_G)$ and imposing suitable boundary and initial conditions. Here we do not consider the mathematical analysis of such a problem, which is far from being trivial. Even in the case of a simple geometry, the well posedeness of the mathematical problem has to be proved.

We conclude that both the mechanical behavior and mass conversion are governed by equations (91)–(93), once the constitutive equations for $\hat{\xi}_G$ and $\hat{\psi}$ are given. A set of constitutive equation for $\hat{\psi}$ and $\hat{\xi}$ is proposed in the forthcoming section.

3 Special constitutive assumptions

In order to provide specific constitutive relations for a growing mixture we assume that entropy production occurs by to mechanisms: internal friction and growth. Friction occurs as a result of the internal solid-liquid momentum exchange and must be somehow related to the relative velocity $\vec{v}_{s,\,f}$. Secondly, we observe that growth in biological systems is an essentially irreversible phenomenon: just a minimum amount of the energy spent for the growth of a soft tissue, for instance, can be recovered under the same form it was supplied to the system. For the sake of simplicity, we therefore assume that growth is a fully irreversible phenomenon and the entropy production rate is obtained defining

$$\hat{\xi} = \beta \mathbf{L}_G \cdot \mathbf{L}_G + H \ \vec{v}_{s,\,f} \cdot \vec{v}_{s,\,f} \,, \tag{95}$$

where β and H are positive constants. We remark that the form (95) is just a candidate that is here adopted to show how calculations can be actually carried out after suitable constitutive assumptions. The specific adopted form is just a possibile choice, suggested the irreversibility of growth, and then mimic the energetic behavior of Newtonian fluids. Real constitutive assumptions for growth remain to be understood.
Equation (77) becomes

$$\begin{aligned}
&\beta\,\mathbf{L}_G \cdot \mathbf{L}_G + H \ \vec{v}_{s,\,f} \cdot \vec{v}_{s,\,f} \ = \\
&\sum_{\alpha=s,f} \nabla \cdot \left[\vec{h}^{\,\alpha} - (\vec{q}^{\alpha} + \rho^{\alpha}\psi^{\alpha}\vec{u}^{\,\alpha})\right] + \\
&\mathrm{tr}\left\{\left[\mathbf{T}^s + \phi^s\lambda - \rho\mathbf{F}_p \frac{\partial\hat{\psi}}{\partial\mathbf{F}_p^T}\right]\mathbf{L}^s\right\} + \\
&\mathrm{tr}\left[\left(\mathbf{T}^f + \lambda\phi^f\mathbf{I}\right)\ \mathbf{L}^f\,\right] + \\
&\mathrm{tr}\left\{\left[\rho\left(\mathbf{F}_p^T\frac{\partial\hat{\psi}}{\partial\mathbf{F}_p} - \frac{\partial\hat{\psi}}{\partial\mathbf{G}}\mathbf{G}^T\right) + \rho^s\lambda\frac{1-\gamma}{\rho_R^f}\mathbf{I}\right]\mathbf{L}_G^T\right\} - \\
&\left[\vec{m}^{\,s} + \frac{\rho^s\vec{v}_{s,\,f}}{2}\,\mathrm{tr}\,\mathbf{L}_G - \gamma\left(\rho_R^s - \rho^s\right)\nabla\hat{\psi} - \lambda\nabla\phi^s\right]\cdot\vec{v}_{s,\,f}.
\end{aligned} \tag{96}$$

from which, because of (78), (79), (80) and (84), we obtain

$$\beta \mathbf{L}_G \cdot \mathbf{L}_G = \operatorname{tr}\left\{ \left[\rho \left(\mathbf{F}_p^T \frac{\partial \hat{\psi}}{\partial \mathbf{F}_p} - \frac{\partial \hat{\psi}}{\partial \mathbf{G}} \mathbf{G}^T \right) + \rho^s \lambda \frac{1-\gamma}{\rho_R^f} \mathbf{I} \right] \mathbf{L}_G^T \right\} \geq 0. \tag{97}$$

Using the maximization of the rate of entropy production criterion we get

$$\mathbf{L}_G = \frac{\mu}{2\beta(\mu+1)} \left[\rho \left(\mathbf{F}_p^T \frac{\partial \hat{\psi}}{\partial \mathbf{F}_p} - \frac{\partial \hat{\psi}}{\partial \mathbf{G}} \mathbf{G}^T \right) + \rho^s \lambda \frac{1-\gamma}{\rho_R^f} \mathbf{I} \right], \tag{98}$$

which substituted in (97) yields $\mu = -2$, so that

$$\frac{D^s \mathbf{G}}{Dt} = \frac{1}{\beta} \left[\rho \left(\mathbf{F}_p^T \frac{\partial \hat{\psi}}{\partial \mathbf{F}_p} - \frac{\partial \hat{\psi}}{\partial \mathbf{G}} \mathbf{G}^T \right) + \rho^s \lambda \frac{1-\gamma}{\rho_R^f} \mathbf{I} \right] \mathbf{G}. \tag{99}$$

We now consider a specific (but still general) form for the Helmholtz free energy function $\hat{\psi}$, writing the corresponding evolution equation for the growth tensor $\mathbf{G}$. We assume that (see [7], for instance)

$$\hat{\psi} = J_G \, W \left(I_{B_p}, II_{B_p}, III_{B_p} \right), \tag{100}$$

where J_G stand for $\det \mathbf{G}$ and W is a scalar function of the invariants of $\mathbf{B}_p$ (denoted by I_{B_p}, II_{B_p} and III_{B_p}, respectively). According to (100) we are considering a material that, in absence of growth, behaves as a classical nonlinear isotropic iperelastic solid. If growth occurs, the free energy is increased just because of mass increase, enforced by the term J_G. In particular, such a choice ensures that $\hat{\psi}$ is frame indifferent.

The stored energy function is split into the product of two factors which account for both deformation (through an isotropic function of $\mathbf{B}_p$) and volumetric growth (through $\det \mathbf{G}$). Obviously the particular choice of W depends on the material which one deals with and it has to be consistent with the experimental data. We refer the reader to [15] where a large class of constitutive functions W is analized.

In conclusion, if one considers (95) and (100), the evolution equation for $\mathbf{G}$ is the following

$$\frac{D^s \mathbf{G}}{Dt} = \left[\mathbf{f}\ (\mathbf{F}_p, \mathbf{G}, \rho_o^s, \gamma) + \mathbf{g}\,(\mathbf{F}_p, \mathbf{G}, \rho_o^s, \gamma) \right] \mathbf{G}, \tag{101}$$

where

$$\mathbf{f}\ (\mathbf{F}_p, \mathbf{G}, \rho_o^s, \gamma) = \frac{\rho\, J_G}{\beta} \left[\mathbf{F}_p^T \frac{\partial W}{\partial \mathbf{F}_p} - W \mathbf{I} \right], \tag{102}$$

$$\mathbf{g}\ (\mathbf{F}_p, \mathbf{G}, \rho_o^s, \gamma) = \frac{\rho^s \lambda (1-\gamma)}{\beta\, \rho_R^f} \mathbf{I}, \tag{103}$$

are two tensor-valued functions.

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