

Finite-Time Lyapunov Exponents for Products of Random Transformations

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I show how continuous products of random transformations constrained by a generic group structure can be studied by using Iwasawa's decomposition into "angular," "diagonal," and "shear" degrees of freedom. In the case of a Gaussian process a set of variables, adapted to the Iwasawa decomposition and still having a Gaussian distribution, is introduced and used to compute the statistics of the finite-time Lyapunov spectrum of the process. The variables also allow to show the exponential freezing of the "shear" degrees of freedom, which contain information about the Lyapunov eigenvectors.

KEY WORDS: Lyapunov exponents; random matrices; path integral; disordered systems; Iwasawa decomposition.

1. INTRODUCTION

Lyapunov exponents represent the natural way to measure the rapidity of divergence of a set of infinitesimally close points carried along by a chaotic or random flow.^(1, 2, 14) The simplest example is given when a smooth N -dimensional dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

is linearized around a trajectory $\mathbf{x}(t)$ giving the linear equation $\dot{\mathbf{y}} = X(\mathbf{x}) \mathbf{y}$ for the separation $\mathbf{y}(t)$ between two infinitesimally close orbits. The generator of the tangent dynamics, $X_{ij} = \partial_i F / \partial x_j$, is known as the stability matrix of the system. The evolution of the initial separation $\mathbf{y}(0)$ is formally given by the expression

$$\mathbf{y}(t) = g(t) \mathbf{y}(0)$$

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where $g(t)$ is the time-ordered product

$$g(t) = T \exp \left[\int_0^t X(t') dt' \right] \quad (1)$$

The N time-dependent Lyapunov exponents $\lambda_1(t) \geq \dots \geq \lambda_N(t)$ are defined as the eigenvalues of the symmetric matrix $(2t)^{-1} \log[g^\dagger(t) g(t)]$, and measure the rate of growth of infinitesimal k -dimensional volumes ($k = 1, \dots, N$). Oseledec⁽³²⁾ proved under very general conditions the existence of the $t \rightarrow \infty$ limits of the $\lambda_k(t)$ and of the eigenvectors $\mathbf{f}_k(t)$ of $g^\dagger(t) g(t)$.

A common *ansatz* for strongly chaotic systems assumes that the stability matrices $X(t)$ decorrelate in a characteristic time, shorter with respect to other time scales of the system, and can therefore be treated as a succession of independent linear transformations.⁽¹⁴⁾ Then the separation $\mathbf{y}(t)$ evolves according to the stochastic equation

$$\dot{\mathbf{y}} = X(t) \mathbf{y} \quad (2)$$

After carefully specifying the statistics of the random process $X(t)$, and choosing in particular one of the standard (Itô or Stratonovich) regularization for (2), the averages of functionals $f[\mathbf{y}(t)]$ over the random realizations $\mathbf{y}(t)$ can be written in the form of path integrals:⁽³⁴⁾

$$\langle f[\mathbf{y}(t)] \rangle = \int \mathcal{D}X f[\mathbf{y}(t)] e^{-S[X]} \quad (3)$$

The problem of computing averages of the form (3) is quite general, and appears both in the context of quantum and classical systems.

In the theory of quasi one-dimensional disordered wires⁽³⁾ the study of low-temperature conductance fluctuations in small metallic samples is reduced to the computation of the statistics of $g^\dagger(L) g(L)$, where $g(L)$ is the $2n \times 2n$ transfer matrix for a wire of length L and n is the number of transverse modes at the Fermi level.⁽¹⁵⁾ The transfer matrix $g(L)$ can be naturally written in the form (1) as an ordered product of transfer matrices for infinitesimal portions of wire. Here the role of time is played by the length L of the sample, and the role of the Lyapunov exponents by (inverse) localization lengths. Ensemble averaging implies the computation of path integrals of the form (3), where the infinitesimal transfer matrices X are constrained by symmetry considerations. In the presence of time reversal symmetry the X matrices belong to the $\mathfrak{sp}(n, \mathbf{R})$ algebra, if time reversal symmetry is broken they belong to $\mathfrak{su}(n, n)$, and in the presence of spin-orbit scattering to $\mathfrak{so}^*(4n)$.⁽²⁵⁾

In the theory of passive scalar transport in n -dimensional space⁽¹⁷⁾ the computation of many statistical quantities, such as the exact p.d.f.'s of the scalar^(7, 11, 13, 20) and of its gradients,^(10, 21) can be reduced to the form (3), where X is a matrix of velocity gradients of the carrying fluid. The matrices X are the infinitesimal generators of the dynamics of an incompressible fluid element and belong to the $\mathfrak{sl}(n, \mathbf{R})$ algebra.

In the study of deterministic chaos, products of random matrices have been used to mimic the chaoticity of dynamical systems.⁽¹⁴⁾ In the case of n -dimensional Hamiltonian systems one has products of random symplectic maps.^(5, 33) This formally leads to expressions of the form (3) where the matrices X belong to the $\mathfrak{sp}(n, \mathbf{R})$ algebra. Similar models have been recently used to study the hydrodynamic modes in the Lyapunov spectrum of extended dynamical systems.⁽¹⁶⁾

In many of these cases analytic results were obtained. Such results were made possible by the presence of three main ingredients: (i) Markovianity;² (ii) Gaussianity; and (iii) the presence of a group structure in the set of transformations and of a statistics compatible with it (i.e., a statistics that can be expressed in terms of the intrinsic geometry of the transformation group).

While the physical support for the first two assumptions is in some cases questionable, the existence of a group structure lays on stronger foundations, since it directly reflects the symmetries of the underlying physical problem.

In the case of quasi one-dimensional wires, results were obtained studying the Fokker–Planck equation for the transmission eigenvalues as a diffusion equation over Riemannian manifolds.^(4, 9) A similar approach has been also applied to the case of passive scalar transport.⁽⁷⁾ On the other hand, functional integral representations of the form (3) for the probability densities can be exploited directly^(8, 10–13, 21, 31) and can provide an intuitive description of the statistics of the Lagrangian trajectories described by Eq. (2). Path integral representations also provide the ground for several approximations, like instanton calculus and perturbative expansions. Actually, the two approaches are complementary and one should switch from one to the other depending on convenience. However, the Lagrangian theory has been worked out mainly on specific examples, often relying on particular features of the given algebras,^(20, 21) and not covering the most general case. The purpose of this work is to present this theory in a sufficiently general setting. It turns out that a formal exposition not only allows to cover many different cases in a unified way, but also unveils the simple

² Although results were obtained also with time-correlated statistics, see, e.g., refs. 11 and 18.

geometric meaning of some formulæ and presents a very simple and appealing scenario, which can hopefully serve as a basis for further investigations.

The starting point of the Lagrangian theory is a convenient decomposition of the time-ordered exponential (1). Let us remind here some facts about the standard method⁽⁶⁾ for the numerical computations of Lyapunov exponents. An orthonormal set of k vectors is evolved for a time t and a Gram–Schmidt orthogonalization procedure is applied to the evolved vectors in order to obtain a new orthonormal set. The procedure is iterated, and by performing an average of the logarithmic k -volume increase over many steps one obtains the approximated sum of the first k Lyapunov exponents. When the evolution operator $g(t)$ belongs to the $SL(n, \mathbf{R})$ group it is easy to see⁽²³⁾ that the Gram–Schmidt orthogonalization procedure is equivalent to the decomposition of the evolution operator in the product of three consecutive transformations, corresponding to a shear, a dilation, and a rotation: the rotation carries the old orthonormal set in the new one, the dilation is responsible for the volume increase, and the shear part inherits from the Gram–Schmidt procedure a triangular form. From an algebraic point of view, the group element $g(t)$ is decomposed into the product of a compact, Abelian and nilpotent part. Iwasawa⁽²⁶⁾ showed that a similar decomposition can be defined for a large class of groups and is a smooth mapping (see ref. 23 and Appendix A).

The fact that emerged from a series of works^(13, 20, 21, 29, 30) is that starting from normally distributed, centered random variables X it is possible in certain cases to introduce new variables, adapted to the Iwasawa decomposition,³ *which are still normally distributed*. Here it is shown that the new variables can be introduced in the quite general case where the random variables X belong to the algebra of a real semisimple Lie group, and their covariance has a form compatible with the group structure (cf. Eq. (5)). It turns out that the definition of the new variables (the $\{A, M\}$ variables defined by Eqs. (13) and (23)) is most easily given in terms of the group structure of the set of transformations, and in particular of the “adjoint operator” $Ad(g) X = gXg^{-1}$.

The fact that the new variables are normally distributed is non trivial, since the Iwasawa transformation is nonlinear, and it is of course quite relevant in practical calculations. In particular, it allows the rapid computation of the Lyapunov spectrum, which is encoded in the statistics of the Abelian part of the Iwasawa decomposition. Although the original variables X are centered, the Abelian part acquires a positive mean value thanks to the non-trivial Jacobian factor associated with the variable

³ Or to the Gauss decomposition, which was considered in refs. 13 and 29 respectively for the $su(2)$ and $su(n)$ cases.

transformation (cf. (35)). This mean value is precisely the finite-time Lyapunov spectrum, which can be expressed explicitly in terms of the structure constants of the algebra of the X variables (cf. Eqs. (38) and (39)).

The new variables however contain more information than just the statistics of Lyapunov exponents. For instance, Eqs. (42), (38), and (37) prove in a quite general setting the freezing of the “shear” (nilpotent) degrees of freedom which was used in refs. 10 and 21 to compute the statistics of passive scalar dissipation. The nilpotent degrees of freedom actually encode information about the Lyapunov eigenvectors $\mathbf{f}_k(t)$. For instance, in the $SL(n, \mathbf{R})$ case it is easy to see that the $\mathbf{f}_k(t)$ can be obtained (up to exponentially small terms) by Gram–Schmidt orthogonalization of the row vectors of the (nilpotent) shear matrix. The exponential freezing of the shear part therefore implies the exponentially rapid convergence of the eigenvectors $\mathbf{f}_k(t)$ to their limits for $t \rightarrow \infty$, thus providing a particularly clear realization of Oseledec theorem.

The work is organized as follows. In Section 2, I define the random process $X(t)$ by assigning its covariance in terms of intrinsic geometric features of the corresponding algebra. This is equivalent to assigning a Feynman–Kac measure on the space of paths $t \rightarrow X(t)$. The multiplicative process $g(t)$ is then defined as a time-ordered exponential through the appropriate (Stratonovich) regularization. In Sections 3–5 the new variables are defined through the following steps: (i) performing the Iwasawa decomposition of the process $g(t)$; (ii) introducing intermediate variables, adapted to the Iwasawa decomposition; (iii) rewriting the Gibbs weight $\exp(-S)$ in terms of the new variables; (iv) computing the Jacobian factor of the variable transformation, i.e., the transformation rule for the “volume form” $\mathcal{D}X$; (v) performing a Lie algebra transformation that leads to new, normally distributed variables. Although the intermediate steps are technical, the final results are simple and are summarized by formulæ (36)–(40). In Section 6 the new variables are used to compute the statistics of the finite time Lyapunov spectrum of a generic linear representation of the process $g(t)$. In Section 7 the cases of the $\mathfrak{sl}(n, \mathbf{R})$ and $\mathfrak{sp}(n, \mathbf{R})$ algebras are worked out in detail. A brief summary of results of Lie algebra theory are given in the Appendix.

2. RANDOM WALKS ON LIE GROUPS

The computational scheme outlined in the Introduction can be implemented in the quite general case where the set of transformations $g(t)$ form a real semisimple Lie group G (this comprises the case of complex groups, as soon as they are seen as real groups endowed with a complex structure). The infinitesimal generators $X(t)$ then lie in the corresponding

algebra \mathfrak{g} . Each real semisimple Lie algebra is characterized by two natural structures: a canonical quadratic form $B(X, Y)$ known as the Killing form, and an involution θ such that $B(X, \theta Y)$ becomes strictly negative definite (see Appendix A). For instance, in the case $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{R})$ (the algebra of real $n \times n$ real matrices with null trace) one has $B(X, Y) = 2n \operatorname{Tr} XY$, $\theta X = -X^t$ and the effect of θ is that of changing the sign of the symmetric part of X .

In order to define a Gaussian probability density on \mathfrak{g} one needs a positive definite quadratic form. I will consider here the general linear combination

$$F(X, Y) = \mu B(X, Y) + \nu B(X, \theta Y) \quad (4)$$

where μ, ν are real parameters such that F be positive definite (see (15) later).

Let $X(t)$ be the Gaussian process on $[0, T]$ with values in \mathfrak{g} , zero mean and covariance

$$\langle X(s) \otimes X(t) \rangle = \delta(t-s) \cdot C \quad (5)$$

where $C = F^{-1}$, i.e., the random process defined by the Feynman–Kac measure

$$d\mu(X) = K \exp \left[-\frac{1}{2} \int_0^T F(X(t), X(t)) dt \right] \prod_{t=0}^T dX(t) \quad (6)$$

where dX is a Euclidean measure on \mathfrak{g} and K is an (infinite) normalization constant. The continuous process $X(t)$ can be seen^(19, 22) as the limit of the discrete process X_j , where $X_j = X(t_j)$, $t_j = j\epsilon$, $\epsilon = T/N$, defined by the probability measure

$$d\mu_N(X) = K_N \exp \left[-\frac{1}{2} \sum_{j=1}^N \epsilon F(X_j, X_j) \right] \prod_{j=1}^N dX_j \quad (7)$$

as $N \rightarrow \infty$.

Let $g(t)$ be the time-ordered exponential defined by

$$g(t) = \mathbf{T} \exp \left(\int_0^t X(\tau) d\tau \right) \equiv \lim_{N \rightarrow \infty} g_N$$

where g_j is defined recursively by

$$g_j = \exp(\epsilon X_j) g_{j-1}, \quad g_0 = e \text{ (the identity)} \quad (8)$$

This discretization prescription is compatible with the group structure, and is of Stratonovich type. As a matter of fact, if G is a linear group (and, in general, on the universal enveloping algebra of \mathfrak{g}) one can compute

$$\begin{aligned} \frac{1}{2\epsilon} (g_j - g_{j-1})(g_j^{-1} + g_{j-1}^{-1}) &= \frac{1}{2\epsilon} (g_j g_{j-1}^{-1} - g_{j-1} g_j^{-1}) \\ &= \frac{1}{2\epsilon} [\exp(\epsilon X_j) - \exp(-\epsilon X_j)] \\ &= X_j + O(\epsilon^2 X_j^3) \end{aligned}$$

obtaining

$$X_j = \frac{g_j - g_{j-1}}{\epsilon} \frac{g_j^{-1} + g_{j-1}^{-1}}{2} + O(\epsilon^2 X_j^3) \quad (9)$$

where $O(\epsilon^2 X_j^3)$ denotes a series of terms having order at least 2 in ϵ and at least 3 in X_j . Such terms can be neglected in averages when $N \rightarrow \infty$, as usual in the theory of Feynman-Kac integrals,⁽¹⁹⁾ since $\langle X_j \otimes X_l \rangle = \epsilon^{-1} \delta_{jl} \cdot C \sim 1/\epsilon$. In other words, in the limit $N \rightarrow \infty$ one can consider

$$X(t) = \dot{g}(t) g^{-1}(t)$$

understanding that the midpoint regularization (9) has to be used throughout.

3. DECOMPOSITION OF THE TRAJECTORIES

Following the general scheme outlined in the Introduction, let us perform the Iwasawa decomposition (56) and (57) of the group element g_j corresponding to the discrete time j in the product of a compact, Abelian and nilpotent part:

$$g_j = k_j a_j n_j \quad (10)$$

where g_j is defined recursively as in (8) and k_j, a_j, n_j are functions of g_j obtained by inverting the diffeomorphism (58). Moreover, let us define K_j, A_j, N_j through the relations

$$k_j = \exp(\epsilon K_j) k_{j-1}, \quad a_j = \exp(\epsilon A_j) a_{j-1}, \quad n_j = \exp(\epsilon N_j) n_{j-1} \quad (11)$$

The variables K, A, N are an intermediate set of new variables, adapted to the Iwasawa decomposition (10). In order to obtain the explicit form of the

variable transformation $X \rightarrow \{K, A, N\}$, one substitutes (10) in (8) and uses (11):

$$\begin{aligned} \exp(\epsilon X_j) &= g_j g_{j-1}^{-1} \\ &= \exp(\epsilon K_j) \cdot \text{Ad}(k_{j-1}) \exp(\epsilon A_j) \cdot \text{Ad}(k_{j-1} a_{j-1}) \exp(\epsilon N_j) \end{aligned}$$

where $\text{Ad}(g) X = g X g^{-1}$. Using (9) one obtains:

$$\begin{aligned} X_j &= \frac{1}{2\epsilon} [\exp(\epsilon X_j) - \exp(-\epsilon X_j)] + O(\epsilon^2 X_j^3) \\ &= K_j + \text{Ad}(k_{j-1}) A_j + \text{Ad}(k_{j-1} a_{j-1}) N_j \\ &\quad + \frac{\epsilon}{2} [K_j, \text{Ad}(k_{j-1}) A_j] + \frac{\epsilon}{2} [K_j, \text{Ad}(k_{j-1} a_{j-1}) N_j] \\ &\quad + \frac{\epsilon}{2} [\text{Ad}(k_{j-1}) A_j, \text{Ad}(k_{j-1} a_{j-1}) N_j] + O(\epsilon^2 X_j^3) \end{aligned} \quad (12)$$

where $O(\epsilon^2 X_j^3)$ denotes a series of terms of at least second order in ϵ and at least third order in X_j, K_j, A_j, N_j . Using

$$\text{Ad} \left(\exp \left(\frac{\epsilon}{2} X \right) \right) Y = Y + \frac{\epsilon}{2} [X, Y] + O(\epsilon^2 X^2 Y)$$

this can be rewritten as

$$X_j = K_j + \text{Ad}(\tilde{k}_{j-1}) A_j + \text{Ad}(\tilde{k}_{j-1} \tilde{a}_{j-1}) N_j + O(\epsilon^2 X_j^3) \quad (13)$$

with $\tilde{k}_{j-1} = \exp(\frac{\epsilon}{2} K_j) k_{j-1}$, $\tilde{a}_{j-1} = \exp(\frac{\epsilon}{2} A_j) a_{j-1}$.

Formula (13) extends the Iwasawa decomposition on the Lie algebra \mathfrak{g} to a natural decomposition

$$(K_j, A_j, N_j) \rightarrow X_j$$

of the random process X_j , which corresponds to the decomposition (10) at the group level. Observe that

$$\tilde{k}_{j-1} = \tilde{k}_{j-1}(K_j, K_{j-1}, \dots, K_1), \quad \tilde{a}_{j-1} = \tilde{a}_{j-1}(A_j, A_{j-1}, \dots, A_1)$$

and that the appearance of \tilde{k}_{j-1} and \tilde{a}_{j-1} means that at the continuum level the midpoint regularization (9) applies also to the group terms arising from the Iwasawa decomposition.

4. DECOMPOSITION OF THE GIBBS WEIGHT

Relation (13) can now be used to express the Gibbs weight defined through (4) in terms of the new variables. Using the orthogonality with respect to B of \mathfrak{a} to both \mathfrak{k} and \mathfrak{n} , of \mathfrak{n} with itself, and properties (53)–(55), a simple computation shows that

$$F(X_j, X_j) = (\mu + \nu) B(K_j - K_j^0, K_j - K_j^0) + (\mu - \nu) B(A_j, A_j) - \frac{\mu - \nu}{2} B(\text{Ad}(\tilde{a}_{j-1}^2) N_j, \theta N_j) \quad (14)$$

with

$$K_j^0 = -\frac{1}{2} [\text{Ad}(\tilde{k}_{j-1} \tilde{a}_{j-1}) N_j + \theta \text{Ad}(\tilde{k}_{j-1} \tilde{a}_{j-1}) N_j]$$

Recalling that $B(X, X) > 0$ on \mathfrak{p} , $B(X, X) < 0$ on \mathfrak{k} , and $B(X, \theta X) < 0$ on \mathfrak{g} , and using (54) and (55), it is clear that $F(X, X) > 0$ if

$$\mu > \nu \quad \text{and} \quad \mu < -\nu \quad (15)$$

The last two rows of (14) can be written more explicitly by expanding over the basis Y_α defined in (62) and a basis H_l of \mathfrak{a} :

$$N_j = \sum_{\alpha \in P} N_j^\alpha Y_\alpha, \quad A_j = \sum_l A_j^l H_l \quad (16)$$

Then using

$$\text{ad}(A_j) Y_\alpha = \alpha(A_j) Y_\alpha \quad (17)$$

$$\text{Ad}(\tilde{a}_{j-1}) Y_\alpha = \exp[\alpha(\exp^{-1} \tilde{a}_{j-1})] Y_\alpha \quad (18)$$

with

$$\exp^{-1} \tilde{a}_{j-1} = \sum_{l=1}^{j-1} \epsilon_l A_l + \frac{\epsilon}{2} A_j \quad (19)$$

one gets

$$B(A_j, A_j) = \sum_{k,l} A_j^k A_j^l B(H_k, H_l) \quad (20)$$

$$B(\text{Ad}(\tilde{a}_{j-1}^2) N_j, \theta N_j) = \sum_{\alpha, \beta \in P} N_j^\alpha N_j^\beta \exp[2\alpha(\exp^{-1} \tilde{a}_{j-1})] B(Y_\alpha, \theta Y_\beta) \quad (21)$$

Here the matrix $B(H_k, H_l)$ is directly related to the Cartan matrix which characterizes the Lie algebra \mathfrak{g} . A list of Cartan matrices for all semisimple Lie algebras can be found in ref. 23. The matrix $B(Y_\alpha, \theta Y_\beta)$ is non-degenerate, it is diagonal if the multiplicities (59) are all 1 and is related to the way the Cartan involution θ permutes the roots otherwise.⁽²³⁾

It is often necessary to compute averages of functionals $F(\{a_j\})$ which do not depend on k_j, n_j . In this case, the form of (14) shows that integration over $\prod_{j=1}^N dK_j$ produces only a constant factor. Integration over $\prod_{j=1}^N dN_j$ produces instead a factor proportional to

$$\begin{aligned} D_N^{-1/2} &\equiv \prod_{j=1}^N \prod_{\alpha \in P} \exp[-\alpha(\exp^{-1} \tilde{a}_{j-1})] \\ &= \prod_{j=1}^N \exp[-2\rho(\exp^{-1} \tilde{a}_{j-1})] \\ &= \prod_{j=1}^N \exp \left[-2 \sum_{l=1}^{j-1} \epsilon\rho(A_l) - \epsilon\rho(A_j) \right] \end{aligned} \quad (22)$$

where notation (65) was used.

Expression (14) takes a simpler form if one introduces the “rotated” variables

$$M_j = \text{Ad}(\tilde{a}_{j-1}) N_j \quad (23)$$

The variable change $N \rightarrow M$ introduces however a Jacobian factor

$$\begin{aligned} \frac{\partial(\{M_j\})}{\partial(\{N_j\})} &= \prod_{j=1}^N \det(\text{Ad}(\tilde{a}_{j-1})|_n) \\ &= \prod_{j=1}^N \exp[\text{Tr ad}(\exp^{-1} \tilde{a}_{j-1})|_n] \\ &= \prod_{j=1}^N \exp \left[\sum_{\alpha \in P} \alpha(\exp^{-1} \tilde{a}_{j-1}) \right] \\ &= \prod_{j=1}^N \exp \left[2 \sum_{l=1}^{j-1} \epsilon\rho(A_l) + \epsilon\rho(A_j) \right] = D_N^{1/2} \end{aligned} \quad (24)$$

Integration over the variables M_j would then produce a constant (independent from A_j) factor. Moreover, factor (24) will be seen to cancel exactly a corresponding factor emerging in the global Jacobian (30).

5. DECOMPOSITION OF THE VOLUME FORM

Along with the variables X_j and K_j, A_j, N_j , which belong to the algebra \mathfrak{g} of infinitesimal transformations, one can consider the “integrated” variables g_j and k_j, a_j, n_j defined in (8) and (10). All together, one has the following commutative diagram of variable transformations:

$$\begin{array}{ccc}
 \{g_j\} & \longrightarrow & \{k_j, a_j, n_j\} \\
 \downarrow & & \downarrow \\
 \{X_j\} & \longrightarrow & \{K_j, A_j, N_j\}
 \end{array} \tag{25}$$

where the variables of the upper row belong to the group G and those of the lower row to the corresponding algebra \mathfrak{g} . The relation between the two set of variables can be found by varying (12):

$$\begin{aligned}
 \delta X_j &= \delta K_j + \text{Ad}(k_{j-1}) \delta A_j + \text{Ad}(k_{j-1} a_{j-1}) \delta N_j \\
 &+ \frac{\epsilon}{2} \{[\delta K_j, \text{Ad}(k_{j-1}) A_j] + [K_j, \text{Ad}(k_{j-1}) \delta A_j] \\
 &+ [\delta K_j, \text{Ad}(k_{j-1} a_{j-1}) N_j] + [K_j, \text{Ad}(k_{j-1} a_{j-1}) \delta N_j] \\
 &+ [\text{Ad}(k_{j-1}) \delta A_j, \text{Ad}(k_{j-1} a_{j-1}) N_j] \\
 &+ [\text{Ad}(k_{j-1}) A_j, \text{Ad}(k_{j-1} a_{j-1}) \delta N_j]\} \\
 &+ O(\epsilon^2 X_j^2 \delta X_j) + f(\delta X_{j-1}, \delta X_{j-2}, \dots, \delta X_1)
 \end{aligned} \tag{26}$$

The variation $\delta X_j \in \text{Tg}$ can be identified with the 1-form $\delta X_j(\cdot) = F(\delta X_j, \cdot)$. In (26), $O(\epsilon^2 X_j^2 \delta X_j)$ denotes a 1-form in $\delta K_j, \delta A_j, \delta N_j$ whose coefficients are at least of second order in ϵ and in K_j, A_j, N_j , while $f(\delta X_{j-1}, \delta X_{j-2}, \dots, \delta X_1)$ denotes a 1-form in $\delta K_l, \delta A_l, \delta N_l$ for $l < j$.

On the other hand, the measure $d\mu(X)$ can be re-expressed in terms of the Haar measure on G , by computing the variation

$$\begin{aligned}
 \delta g_j g_j^{-1} &= \left[\epsilon \delta X_j + \frac{\epsilon^2}{2} (\delta X_j X_j + X_j \delta X_j) + O(\epsilon^3 X_j^2 \delta X_j) \right] \cdot g_{j-1} g_j^{-1} \\
 &+ f(\delta X_{j-1}, \dots, \delta X_1) \\
 &= \left[\epsilon \delta X_j + \frac{\epsilon^2}{2} (\delta X_j X_j + X_j \delta X_j) \right] (1 - \epsilon X_j) \\
 &+ O(\epsilon^3 X_j^2 \delta X_j) + f(\delta X_{j-1}, \dots, \delta X_1) \\
 &= \epsilon \delta X_j + \frac{\epsilon^2}{2} [X_j, \delta X_j] + O(\epsilon^3 X_j^2 \delta X_j) + f(\delta X_{j-1}, \dots, \delta X_1)
 \end{aligned}$$

so that

$$\frac{1}{\epsilon} \delta g_j g_j^{-1} = \text{Ad} \left(\exp \left(\frac{\epsilon}{2} X_j \right) \right) \delta X_j + O(\epsilon^2 X_j^2 \delta X_j) + f(\delta X_{j-1}, \dots, \delta X_1)$$

In the continuum limit, terms $O(\epsilon^2 X_j^2 \delta X_j)$ contribute to the measure with a constant factor which ensures the correct normalization.⁴ One then has

$$\bigwedge_{j=1}^N \bigwedge_{l=1}^{\dim \mathfrak{g}} \delta_l X_j \propto \bigwedge_{j=1}^N \bigwedge_{l=1}^{\dim G} \delta_l g_j g_j^{-1}$$

where the $\delta_l X$ span $T_0 \mathfrak{g}$, $\delta_l g g^{-1}$ span $T_e G \simeq T_0 \mathfrak{g}$, and one uses the facts that

1. the $f(\delta X_{j-1}, \dots, \delta X_1)$ terms cancel in the exterior product with δX_j terms;
2. semisimple Lie groups are unimodular, i.e., $|\det \text{Ad}(g)| = 1$ for every $g \in G$.

So:

$$\prod_{j=1}^N dX_j \propto \prod_{J=1}^N dg_J$$

where dg denotes the Haar measure on G . In other words, the variable transformation $\{X_j\} \rightarrow \{g_j\}$ has trivial Jacobian and $d\mu(X) = d\mu(g)$, where

$$d\mu(g) = K' \exp \left[-\frac{1}{2} \int_0^T F(\dot{g}g^{-1}, \dot{g}g^{-1}) dt \right] \prod_{t=0}^T dg(t)$$

To the two horizontal arrows of diagram (25) there correspond the Jacobians defined by

$$\prod_{j=1}^N dg_j = J_N^G(\{k_j, a_j, n_j\}) \prod_{j=1}^N dk_j da_j dn_j \tag{27}$$

$$\prod_{j=1}^N dX_j = J_N^{\mathfrak{g}}(\{K_j, A_j, N_j\}) \prod_{j=1}^N dK_j dA_j dN_j \tag{28}$$

⁴ One can treat terms $\sim \epsilon^2 X_j^2 dX_j$ as a perturbation in integrals over the measure (7) and use Wick's theorem to compute their contribution. They give a contribution $\langle \epsilon^2 X_j^2 \rangle \sim \epsilon$, but their product with any power of ϵX_j higher than 0 is negligible in the continuum limit. The contributions $\sim \epsilon$ sum up to give a constant factor.

The variable transformations corresponding to the two vertical arrows have trivial Jacobians, so

$$J_N^g(\{K_j, A_j, N_j\}) \propto J_N^G(\{k_j, a_j, n_j\}) \quad (29)$$

The product of Haar measures on both sides of (27) is invariant under the global transformations

$$g_j \rightarrow k_R g_j, \quad g_j \rightarrow g_j n_L, \quad \text{with } k_R \in K, \quad n_L \in N$$

which correspond to

$$k_j \rightarrow k_R k_j, \quad n_j \rightarrow n_j n_L$$

By induction over N , this shows that J_N^G does not actually depend on $\{k_j, n_j\}$. From (29) then follows that J_N^g does not depend on $\{K_j, N_j\}$.

The actual value of the Jacobian can now be computed either at the group or at the algebra level. At the group level, using (64), one gets

$$\begin{aligned} J_N^G &= \prod_{j=1}^N \exp[2\rho(\exp^{-1} a_j)] = \prod_{j=1}^N \exp\left[2 \sum_{l=1}^j \epsilon \rho(A_l)\right] \\ &= \prod_{j=1}^N \exp\left[2 \sum_{l=1}^{j-1} \epsilon \rho(A_l) + \epsilon \rho(A_j)\right] \cdot \exp\left[\sum_{j=1}^N \epsilon \rho(A_j)\right] \\ &= D_N^{1/2} \cdot J_N^{G, \text{loc}} \end{aligned} \quad (30)$$

The decomposition of J_N^G into an ultralocal part $D_N^{1/2}$ and a local part $J_N^{G, \text{loc}}$ is such that the ultralocal part cancels exactly the factor (22) which appears when integrating functionals of the form $F(\{a_j\})$ over $\prod_{j=1}^N dK_j \cdot \prod_{j=1}^N dN_j$. The ultralocal factor $D_N^{1/2}$ is also exactly canceled when passing to the “rotated” variables M_j defined in Eq. (23).

The result (30) can be checked by repeating the computation at the algebra level. Recalling that J_N^g cannot depend on K_j, N_j , we can set $K_j = 0, N_j = 0$ in (26), getting

$$\begin{aligned} \delta X_j &= \delta K_j + \delta A_j + \text{Ad}(a_{j-1}) \delta N_j + \frac{\epsilon}{2} [\delta K_j, A_j] + \frac{\epsilon}{2} [A_j, \text{Ad}(a_{j-1}) \delta N_j] \\ &\quad + O(\epsilon^2 X_j^2 \delta X_j) + f(\delta X_{j-1}, \dots, \delta X_1) \\ &= \text{Ad}\left(\exp\left(-\frac{\epsilon}{2} A_j\right)\right) \delta K_j + \delta A_j + \text{Ad}\left(\exp\left(\frac{\epsilon}{2} A_j\right) a_{j-1}\right) \delta N_j \\ &\quad + O(\epsilon^2 X_j^2 \delta X_j) + f(\delta X_{j-1}, \dots, \delta X_1) \end{aligned} \quad (31)$$

Using (61) and (62) to expand $N_j = \sum_{\alpha \in P} \delta N_j^\alpha Y_\alpha$, one gets

$$\text{Ad} \left(\exp \left(\frac{\epsilon}{2} A_j \right) a_{j-1} \right) \delta N_j = \sum_{\alpha \in P} \exp \left[\alpha \left(\frac{\epsilon}{2} A_j + \sum_{l=1}^{j-1} \epsilon A_l \right) \right] \delta N_j^\alpha Y_\alpha \quad (32)$$

Using (61) and (62) to expand $\delta K_j = \delta K_j^0 + \sum_{\alpha \in P} \delta K_j^\alpha (Y_\alpha + \theta Y_\alpha)$, where $\delta K_j^0 \in \mathfrak{m}$, and recalling (60), one gets

$$\begin{aligned} \text{ad}(A_j) \delta K_j &= \sum_{\alpha \in P} \alpha(A_j) \delta K_j^\alpha (Y_\alpha - \theta Y_\alpha) \\ &= \sum_{\alpha \in P} \alpha(A_j) \delta K_j^\alpha [2Y_\alpha - (Y_\alpha + \theta Y_\alpha)] \\ &= N_j^0 - \sum_{\alpha \in P} \alpha(A_j) \delta K_j^\alpha (Y_\alpha + \theta Y_\alpha) \end{aligned}$$

with $N_j^0 \in \mathfrak{n}$, and

$$\begin{aligned} &\left(1 - \frac{\epsilon}{2} \text{ad}(A_j) \right) \delta K_j \\ &= \sum_{\alpha \in P} \exp \left[\alpha \left(\frac{\epsilon}{2} A_j \right) \right] \delta K_j^\alpha (Y_\alpha + \theta Y_\alpha) - \frac{\epsilon}{2} N_j^0 + O(\epsilon^2 A_j^2 \delta K_j) \quad (33) \end{aligned}$$

Now using (31)–(33) and the decomposition (56), observing that the terms $f(\delta X_{j-1}, \dots, \delta X_1)$ cancel in the exterior product with δX_j (the transformation $\{X_j\} \rightarrow \{K_j, A_j, N_j\}$ is triangular in the index j), and that $O(\epsilon^2 X_j^2 \delta X_j)$ terms contribute only to a normalization factor (see footnote 3), one gets

$$\begin{aligned} J_N^g &\propto \prod_{j=1}^N \prod_{\alpha \in P} \exp \left[\alpha \left(\sum_{l=1}^{j-1} \epsilon A_l + \frac{\epsilon}{2} A_j \right) \right] \exp \left[\alpha \left(\frac{\epsilon}{2} A_j \right) \right] \\ &= \prod_{j=1}^N \exp \left[2 \sum_{l=1}^{j-1} \epsilon \rho(A_l) + \epsilon \rho(A_j) \right] \cdot \exp \left[\sum_{j=1}^N \epsilon \rho(A_j) \right] \\ &= D^{1/2} \cdot J_N^{g, \text{loc}} \quad (34) \end{aligned}$$

as expected.

Taking into account (30), (34), and (24) it is clear that the Jacobian of the transformation

$$\{X_j\} \rightarrow \{K_j, A_j, M_j\}$$

is proportional to the simple local factor

$$J_N^{g, \text{loc}} = \exp \left[\sum_{j=1}^N \epsilon \rho(A_j) \right] \quad (35)$$

6. STATISTICS OF LYAPUNOV EXPONENTS

Classical groups are groups of transformations of \mathbf{R}^n or \mathbf{C}^n . Let $(,)$ be a scalar product on \mathbf{R}^n or a hermitian product on \mathbf{C}^n . Let \mathbf{v} be a vector of \mathbf{R}^n or \mathbf{C}^n , $g\mathbf{v}$ the action of the group element g on \mathbf{v} . Let $g(t)$ be a random walk on a (real or complex) classical group G . Then

$$\|g(t)\mathbf{v}\|^2 = (g(t)\mathbf{v}, g(t)\mathbf{v}) = (g^\dagger(t)g(t)\mathbf{v}, \mathbf{v})$$

so that the rate of growth of $\|g(t)\mathbf{v}\|$ is related to the highest eigenvalue of $g^\dagger(t)g(t)$. More generally, the rate of growth of the parallelepiped generated by l vectors $\mathbf{v}_1, \dots, \mathbf{v}_l$, is related to the sum of the first l eigenvalues $\exp(2\lambda_1(t)t), \dots, \exp(2\lambda_l(t)t)$ of the hermitian matrix $g^\dagger(t)g(t)$.⁽¹⁴⁾

$$\lambda_1(t) + \dots + \lambda_l(t) = \frac{1}{t} \log \frac{\text{Vol}(g(t)\mathbf{v}_1 \wedge \dots \wedge g(t)\mathbf{v}_l)}{\text{Vol}(\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_l)}$$

The eigenvalues $\lambda_1(t), \dots, \lambda_l(t)$ are the (time dependent) Lyapunov exponents of the process $g(t)$.

For the classical algebras the Cartan involution θ is just $\theta Z = -Z^\dagger$, which exponentiates to the group automorphism Θ defined by $\Theta g = (g^\dagger)^{-1}$. Let T be a finite dimensional representation of G such that (66) holds. In general, the (time dependent) Lyapunov exponents of the process $g(t)$ can be seen as the eigenvalues of

$$(2t)^{-1} \log[T^\dagger(g)T(g)] = -(2t)^{-1} \log T(g^{-1}\Theta g) = (2t)^{-1} \log T(\Theta n^{-1} \cdot a^2 n)$$

or equivalently as the eigenvalues of

$$(2t)^{-1} \log[T(a^2)T(n\Theta n^{-1})]$$

After integrating out the \mathbf{k} component, incorporating the contribution of $J_N^{\mathfrak{g}, \text{loc}}$, and using (20), (21), and (52), $F(X_j, X_j)$ reduces to,

$$\begin{aligned} F^{a+n}(X_j, X_j) &= (\mu - \nu) \sum_{k,l} B(H_k, H_l) A_j^k A_j^l - \sum_l \sum_{\alpha \in P} B(H_l, H_\alpha) A_j^l \\ &\quad - \frac{\mu - \nu}{2} \sum_{\alpha, \beta \in P} \exp \left[\alpha \left(\sum_{l=1}^{j-1} 2\epsilon A_j + \epsilon A_j \right) \right] B(Y_\alpha, \theta Y_\beta) N_j^\alpha N_j^\beta \end{aligned} \quad (36)$$

$$= F^a(X_j, X_j) - \frac{\mu - \nu}{2} \sum_{\alpha, \beta \in P} B(Y_\alpha, \theta Y_\beta) M_j^\alpha M_j^\beta \quad (37)$$

After integrating out the M -variables, and taking into account the observations following (22) and (30), $F^{a+n}(X_j, X_j)$ reduces to $F^a(X_j, X_j)$. This shows that in the $\{K_j, A_j, M_j\}$ representation the A_j are Gaussian random variables with positive mean

$$\langle A_j^k \rangle = \frac{1}{2(\mu - \nu)} \sum_l \Gamma^{kl} \sum_{\alpha \in P} B(H_l, H_\alpha) \quad (38)$$

where

$$(\Gamma^{-1})_{kl} = B(H_k, H_l) \quad (39)$$

and covariance

$$\langle A_j^k A_j^l \rangle_c = \frac{1}{\mu - \nu} \Gamma^{kl} \quad (40)$$

On the other hand, (38) and (40) together with (36) show that the n variables diffuse, while the a variables perform a ballistic motion, diffusing around straight trajectories. This means that for $t \gg 1$ and with logarithmic precision the time dependent Lyapunov exponents coincide with the eigenvalues of

$$\begin{aligned} \frac{1}{2t} \log T(a^2(t)) &= \frac{1}{t} \tau(\exp^{-1}(a(t))) \\ &= \frac{1}{t} \tau \left(\int_0^t A(t') dt' \right) \\ &= \frac{1}{t} \int_0^t dt' \sum_k A^k(t') \tau_k \end{aligned} \quad (41)$$

with $T(\exp(X)) = \exp(\tau(X))$ and $\tau_k = \tau(H_k)$. The operators τ_k all commute, so it is possible to write (41) in block-matrix form. This means that the Lyapunov exponents are just the $\frac{1}{t} \int_0^t dt' A^k$ multiplied by the eigenvalues of the operators τ_k .

Let us describe more precisely the statistics of the n variables. Expression (37) shows that the M_j are Gaussian random variables with zero mean and $O(1)$ covariance. Expanding the M_j and N_j on the basis Y_α one finds the explicit expression

$$N_j^\alpha = \exp \left[-\alpha \left(\sum_{l=1}^{j-1} \epsilon A_l + \frac{\epsilon}{2} A_j \right) \right] M_j^\alpha, \quad \alpha \in P \quad (42)$$

From the positiveness of α and of $\langle A_j^k \rangle$, $k = 1, \dots, \dim \mathfrak{g}$, and passing to the continuum limit, it follows that typical values of the $N^\alpha(t)$ are exponentially small in t . Correspondingly, the $n(t)$ variables become frozen after a time $t = O(1)$. The freezing of the n variables has been observed in ref. 10 and used to obtain analytic results about the statistics of the dissipation of a passive scalar field in refs. 10 and 21. Moreover, it provides a nice realization of Oseledec theorem about the stability of the Lyapunov basis in the $t \rightarrow \infty$ limit.⁽³²⁾

7. EXAMPLES

Let E_{ij} denote a square matrix with entry 1 where the i th row and the j th column meet, all other entries being 0. Then

$$E_{ij}E_{kl} = \delta_{jk}E_{il}, \quad [E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$$

The Case of $\mathfrak{sl}(n, \mathbf{R})$. In this case \mathfrak{g}_C is the algebra of $n \times n$ matrices with zero trace. Letting

$$H_i = E_{ii} - E_{nn} \quad (1 \leq i \leq n-1), \quad \mathfrak{h}_C = \sum_i \mathbf{C}H_i \quad (43)$$

one has the direct decomposition⁽²³⁾

$$\mathfrak{g}_C = \mathfrak{h}_C + \sum_{i \neq j} \mathbf{C}E_{ij} \quad (44)$$

If $H \in \mathfrak{h}_C$ and $e_i(H)$ ($1 \leq i \leq n$) are the diagonal elements of H , one has

$$[H, E_{ij}] = (e_i(H) - e_j(H)) E_{ij} \quad (45)$$

The Killing form is

$$B(X, Y) = 2n \operatorname{Tr}(XY) \quad (46)$$

The roots are

$$e_i - e_j \quad (1 \leq i, j \leq n)$$

which are duals to the elements $\frac{1}{2n}(H_i - H_j) \in \mathfrak{h}_R$. The compact real form for $\mathfrak{sl}(n, \mathbf{C})$ is $\mathfrak{u} = \mathfrak{su}(n)$, the algebra of skew hermitian matrices; \mathfrak{h}_C are the diagonal matrices with null trace, $\mathfrak{a} = \mathfrak{h}_R$ are the real matrices of \mathfrak{h}_C . Taking lexicographic ordering with respect to the basis $e_i - e_n$ ($1 \leq i \leq n-1$) of the

dual of \mathfrak{h}_R , i.e., assuming $e_i - e_n > e_j - e_n$ and consequently $e_i - e_j > 0$ for $i < j$, one finds

$$\mathfrak{n}_C = \sum_{i < j} \mathbf{C}E_{ij} = \{\text{strictly upper triangular matrices}\}$$

$\mathfrak{g} = \mathfrak{sl}(n, \mathbf{R})$ is the real form corresponding to the conjugation $\sigma Z = \bar{Z}$. The Cartan involution is $\theta X = -X^{\text{tr}}$. One has $\mathfrak{k} = \mathfrak{u} \cap \mathfrak{g} = \mathfrak{so}(n)$, \mathfrak{p} are the real symmetric matrices with null trace, \mathfrak{a} the diagonal matrices of \mathfrak{p} , \mathfrak{n} the real matrices of \mathfrak{n}_C . The restricted roots coincide with the roots $e_i - e_j$. Letting $A_j = \sum_{k=1}^{n-1} A_j^k H_k$ and using (38)–(40), (43), and (46), a simple computation gives:

$$\langle A_j^k \rangle = \frac{1}{2(\mu - \nu)} \frac{n - 2k + 1}{2n}, \quad k = 1, \dots, n-1 \quad (47)$$

and

$$\langle A_j^k A_j^l \rangle_c = \frac{1}{2n(\mu - \nu)} \left(\delta^{kl} - \frac{1}{n} \right)$$

Eq. (47) reproduces Eq. (39) in ref. 20 letting $\mu = 1/4Dn$, $\nu = 0$ and Eq. (25) in ref. 21 letting $\mu = 1/2n^2(n+2)D$, $\nu = -(n+1)/2n^2(n+2)D$.

The Case of $\mathfrak{sp}(n, \mathbf{R})$. In this case \mathfrak{g}_C is the algebra of complex matrices of the form $\begin{pmatrix} X & Y \\ Z & -X^{\text{tr}} \end{pmatrix}$, with $Y = Y^{\text{tr}}$ and $Z = Z^{\text{tr}}$. Letting

$$H_i = E_{ii} - E_{n+i, n+i} \quad (1 \leq i \leq n), \quad \mathfrak{h}_C = \sum_{i=1}^n \mathbf{C}H_i \quad (48)$$

one has the direct decomposition⁽²³⁾

$$\begin{aligned} \mathfrak{g}_C = \mathfrak{h}_C + \sum_{i \leq j} \mathbf{C}(E_{n+i, j} + E_{n+j, i}) + \sum_{i \leq j} \mathbf{C}(E_{i, n+j} + E_{j, n+i}) \\ + \sum_{i \neq j} \mathbf{C}(E_{i, j} + E_{n+j, n+i}) \end{aligned} \quad (49)$$

Let e_j be the linear form on \mathfrak{h} given by $e_j(H_i) = \delta_{ij}$; then

$$[H, E_{n+i, j} + E_{n+j, i}] = -(e_i(H) + e_j(H))(E_{n+i, j} + E_{n+j, i}) \quad (i \leq j)$$

$$[H, E_{i, n+j} + E_{j, n+i}] = (e_i(H) + e_j(H))(E_{i, n+j} + E_{j, n+i}) \quad (i \leq j)$$

$$[H, E_{i, j} - E_{n+j, n+i}] = (e_i(H) - e_j(H))(E_{i, j} - E_{n+j, n+i}) \quad (i \neq j)$$

The Killing form is

$$B(X, Y) = (2n+2) \operatorname{Tr}(XY) \quad (50)$$

The non zero roots are (\pm signs read independently)

$$\pm 2e_i \quad (1 \leq i \leq n), \quad \pm e_i \pm e_j \quad (1 \leq i < j \leq n)$$

The roots $2e_j$ are dual to the elements $\frac{2}{4(n+1)} H_i \in \mathfrak{h}_R$. The compact real form is

$$\mathfrak{u} = \mathfrak{sp}(n) = \left\{ \begin{pmatrix} U & V \\ -\bar{V} & \bar{U} \end{pmatrix} : U + U^\dagger = 0, V = V^{\operatorname{tr}} \right\}$$

Taking lexicographic ordering with respect to the basis e_i ($1 \leq i \leq n$) of the dual of \mathfrak{h}_R it is seen that the positive roots are $e_i + e_j$ ($i \leq j$) and $e_i - e_j$ ($i < j$). Consequently

$$\begin{aligned} \mathfrak{n}_C &= \sum_{i \leq j} \mathbf{C}(E_{i, n+j} + E_{j, n+i}) + \sum_{i < j} \mathbf{C}(E_{i, j} - E_{n+j, n+i}) \\ &= \left\{ \begin{pmatrix} X & Y \\ 0 & -X^{\operatorname{tr}} \end{pmatrix}, Y = Y^{\operatorname{tr}}, X \text{ strictly upper triangular} \right\} \end{aligned}$$

$\mathfrak{g} = \mathfrak{sp}(n, \mathbf{R})$ is the real form corresponding to the conjugation $\sigma Z = \bar{Z}$. One has

$$\mathfrak{k} = \mathfrak{sp}(n) \cap \mathfrak{so}(2n)$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} : X = X^{\operatorname{tr}}, X = \bar{X}, Y = Y^{\operatorname{tr}}, Y = \bar{Y} \right\}$$

$$\mathfrak{a} = \{\text{diagonal matrices of } \mathfrak{p}\}$$

$$\mathfrak{n} = \{\text{real matrices of } \mathfrak{n}_C\}$$

Restricted roots just coincide with roots. Letting $A_j = \sum_{k=1}^n A_j^k H_k$ and using (38)–(40), (48), and (50), a simple computation gives

$$\langle A_j^k \rangle = \frac{1}{2(\mu - \nu)} \frac{n - k + 1}{2(n + 1)}, \quad k = 1, \dots, n$$

and

$$\langle A_j^k A_j^l \rangle_c = \frac{1}{4(n+1)(\mu - \nu)} \delta^{kl}$$

APPENDIX A. LIE-CARTAN THEORY

I recall here some results from the theory of Lie algebras^(23, 24, 27, 28) for convenience and in order to establish the notations, which partially parallel those of ref. 23.

Any complex Lie algebra \mathfrak{g}_C can be seen as a real Lie algebra endowed with a complex structure. A real form of a complex Lie algebra \mathfrak{g}_C is a real Lie algebra \mathfrak{g} such that $\mathfrak{g}_C = \mathfrak{g} + i\mathfrak{g}$ (direct sum of vector spaces) where \mathfrak{g} and $i\mathfrak{g}$ are identified with subalgebras of \mathfrak{g}_C . Thus \mathfrak{g}_C is isomorphic to the complexification of \mathfrak{g} . (E.g., $\mathfrak{sl}(n, \mathbf{R})$ is a real form of $\mathfrak{sl}(n, \mathbf{C})$.) There is a 1-1 correspondence between the real forms of a Lie algebra and the conjugations $\sigma: X + iY \rightarrow X - iY$ ($X, Y \in \mathfrak{g}$). (E.g., $\mathfrak{sl}(n, \mathbf{R})$ corresponds to the conjugation $\sigma: Z \rightarrow \bar{Z}$ of $\mathfrak{sl}(n, \mathbf{C})$.) A compact Lie algebra is a real Lie algebra which is isomorphic to the Lie algebra of a compact Lie group. Every semisimple Lie algebra \mathfrak{g}_C over \mathbf{C} has a compact real form \mathfrak{u} , which is unique (up to automorphisms; e.g., $\mathfrak{su}(n)$ is the compact real form of $\mathfrak{sl}(n, \mathbf{C})$); it corresponds to the conjugation $\tau: Z \rightarrow -Z^\dagger$.

A Cartan subalgebra \mathfrak{h}_C of a complex Lie algebra \mathfrak{g}_C is a maximal Abelian subalgebra such that the endomorphism $\text{ad } H: X \rightarrow [H, X]$ is semisimple (i.e., \mathfrak{g}_C can be written as a direct sum of irreducible invariant subspaces of $\text{ad } H$ for all $H \in \mathfrak{h}_C$). Every semisimple Lie algebra over \mathbf{C} has a Cartan subalgebra which is unique (up to automorphisms; e.g., diagonal matrices with null trace form a Cartan subalgebra of $\mathfrak{sl}(n, \mathbf{C})$).

On each real and complex Lie algebra one can define the Killing form $B(X, Y) = \text{Tr}(\text{ad } X \text{ ad } Y)$, where Tr denotes the trace of the vector space endomorphism. (E.g., $B(X, Y) = 2n \text{Tr } XY$ for $X, Y \in \mathfrak{sl}(n, \mathbf{C})$.) Every semisimple Lie algebra over \mathbf{C} has a real form \mathfrak{u} such that B is strictly negative definite on \mathfrak{u} (it then follows that \mathfrak{u} is compact, cf. above; e.g., $\text{Tr } X^2 = -\text{Tr } XX^\dagger < 0$ for $X \in \mathfrak{su}(n)$, $X \neq 0$).

A complex Lie algebra \mathfrak{g}_C over \mathbf{C} can be decomposed in the direct sum of the simultaneous eigenspaces (root subspaces)

$$\mathfrak{g}_C^\alpha = \{X \in \mathfrak{g}_C : \text{ad}(H) X = \alpha(H) X \text{ for all } H \in \mathfrak{h}_C\}$$

where the linear function $\alpha: \mathfrak{h}_C \rightarrow \mathbf{C}$ is called a root (of \mathfrak{g}_C with respect to \mathfrak{h}_C) if $\mathfrak{g}_C^\alpha \neq \{0\}$. Let Δ denote the set of nonzero roots. One has the root decomposition

$$\mathfrak{g}_C = \mathfrak{h}_C + \sum_{\alpha \in \Delta} \mathfrak{g}_C^\alpha \quad (\text{direct sum}) \quad (51)$$

with the properties: (i) $\dim \mathfrak{g}_C^\alpha = 1$ for each $\alpha \in \Delta$; (ii) \mathfrak{g}_C^α and \mathfrak{g}_C^β are orthogonal under B if α, β are any two roots and $\beta \neq -\alpha$; (iii) B is non

degenerate on \mathfrak{h}_C and for each linear function α on \mathfrak{h}_C there exists a unique $H_\alpha \in \mathfrak{h}_C$ such that

$$\alpha(H) = B(H, H_\alpha) \quad (52)$$

for all $H \in \mathfrak{h}_C$; (iv) if $\alpha \in \Delta$, then $-\alpha \in \Delta$ and $[\mathfrak{g}_C^\alpha, \mathfrak{g}_C^{-\alpha}] = \mathbb{C}H_\alpha$ and $\alpha(H_\alpha) \neq 0$. Let $\mathfrak{h}_R = \sum_{\alpha \in \Delta} \mathbb{R}H_\alpha$. Then by (iii), Δ can be seen as a subset of the dual space \mathfrak{h}'_R . Given any basis on \mathfrak{h}_R one can introduce a lexicographic order on \mathfrak{h}'_R and thus on Δ . This gives the concept of positive and negative roots. A positive root is called simple if it cannot be written as the sum of two positive roots. There are exactly $\dim \mathfrak{h}_R$ simple roots α_j and each $\alpha \in \Delta$ can be written as $\sum n_j \alpha_j$ with n_j integers which are either all positive or all negative.

For each $\alpha \in \Delta$ one can choose $X_\alpha \in \mathfrak{g}_C^\alpha$ such that for all $\alpha, \beta \in \Delta$:

$$[X_\alpha, X_{-\alpha}] = H_\alpha$$

$$[H, X_\alpha] = \alpha(H) X_\alpha \quad \text{for } H \in \mathfrak{h}_C$$

$$[X_\alpha, X_\beta] = 0 \quad \text{if } \alpha \neq -\beta \quad \text{and} \quad \alpha + \beta \notin \Delta$$

$$[X_\alpha, X_\beta] = N_{\alpha, \beta} X_{\alpha + \beta} \quad \text{if } \alpha + \beta \in \Delta, \quad \text{with } N_{\alpha, \beta} = -N_{-\alpha, -\beta}$$

$$B(X_\alpha, X_\beta) = \delta_{\alpha + \beta, 0}$$

On the real algebra \mathfrak{u} generated by the elements $iH_\alpha, X_\alpha - X_{-\alpha}, i(X_\alpha + X_{-\alpha})$, $\alpha \in \Delta$, the form B is strictly negative definite; so, \mathfrak{u} is a compact real form of \mathfrak{g}_C (cf. above). If τ is the conjugation of \mathfrak{g}_C with respect to \mathfrak{u} , $\tau X_\alpha = -X_{-\alpha}$ for $\alpha \in \Delta$. The set $\{X_\alpha\}$ is called a Weyl basis.

A Cartan decomposition of a semisimple Lie algebra \mathfrak{g} over \mathbb{R} is a direct sum $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ in a subalgebra \mathfrak{k} and a vector subspace \mathfrak{p} such that the mapping $\theta: K + P \rightarrow K - P$ ($K \in \mathfrak{k}, P \in \mathfrak{p}$) is an automorphism of \mathfrak{g} and the form $-B(X, \theta Y)$ is strictly positive definite. θ is called a Cartan involution. Each semisimple Lie algebra over \mathbb{R} has a Cartan decomposition. If \mathfrak{g}_C is the complexification of \mathfrak{g} , σ is the conjugation of \mathfrak{g}_C with respect to \mathfrak{g} , one can always choose a real form \mathfrak{u} such that $\sigma \mathfrak{u} \subset \mathfrak{u}$, so that $\mathfrak{k} = \mathfrak{u} \cap \mathfrak{g}$, $\mathfrak{p} = i\mathfrak{u} \cap \mathfrak{g}$, and $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$; \mathfrak{u} is then a compact real form of \mathfrak{g}_C . (E.g., for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ the Cartan decomposition is the decomposition of a matrix into its antisymmetric and symmetric part; in this case $\theta X = -{}^t X$, $\mathfrak{k} = \mathfrak{so}(n)$, $\mathfrak{p} = \{\text{symmetric matrices}\}$, and $\mathfrak{k} + i\mathfrak{p} = \mathfrak{su}(n)$.) If \mathfrak{g}_C is a semisimple Lie algebra over \mathbb{C} , \mathfrak{u} its compact real form, then $\mathfrak{g}_C = \mathfrak{u} + i\mathfrak{u}$ is a Cartan decomposition of \mathfrak{g}_C . (E.g., for $\mathfrak{g}_C = \mathfrak{sl}(n, \mathbb{C})$ it is the decomposition of a matrix into its skew-hermitian and hermitian part.)

Since

$$\theta = 1 \quad \text{on } \mathfrak{k} \quad \text{and} \quad \theta = -1 \quad \text{on } \mathfrak{p} \quad (53)$$

Then, for all $x, y \in \mathfrak{g}$

$$\theta \operatorname{ad}(X) Y = \operatorname{ad}(\theta X) \theta Y \quad (54)$$

$$\theta \operatorname{Ad}(\exp(X)) Y = \operatorname{Ad}(\exp(\theta X)) \theta Y \quad (55)$$

where Ad is the adjoint representation of G on \mathfrak{g} (e.g., $\operatorname{Ad}(g) X = gXg^{-1}$ if G, \mathfrak{g} are linear), so that $\operatorname{Ad}(\exp(X)) = \exp(\operatorname{ad}(X))$.

Let \mathfrak{h} be a maximal Abelian subalgebra of \mathfrak{g} , with $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$ as large as possible, and let $\mathfrak{a}_1 = \mathfrak{h} \cap \mathfrak{k}$. Then $\mathfrak{h}_R = \mathfrak{a} + i\mathfrak{a}_1$ and $\mathfrak{h}_C = \mathfrak{h}_R + i\mathfrak{h}_R$ is a Cartan subalgebra of \mathfrak{g}_C . (E.g., for $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{R})$, $\mathfrak{a}_1 = \{0\}$, and $\mathfrak{a} = \mathfrak{h}_R$ are the real diagonal matrices with null trace.) Let P be the set of positive roots that do not vanish identically on \mathfrak{a} (for $\mathfrak{sl}(n, \mathbf{R})$ this just coincides with the set of positive roots). Let

$$\mathfrak{n}_C = \sum_{\alpha \in P} \mathfrak{g}_C^\alpha, \quad \mathfrak{n} = \mathfrak{n}_C \cap \mathfrak{g}$$

Then \mathfrak{n}_C and \mathfrak{n} are nilpotent Lie algebras and one has the direct vector space sum

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n} \quad (56)$$

(Iwasawa decomposition). From (ii) it follows that \mathfrak{a} is orthogonal with respect to B to both \mathfrak{k} and \mathfrak{n} , and that \mathfrak{n} is orthogonal to itself. The decomposition (56) exponentiates to

$$G = KAN \quad (57)$$

where $K = \exp \mathfrak{k}$, $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$, i.e., the mapping

$$(k, a, n) \rightarrow kan \quad (58)$$

is a diffeomorphism of $K \times A \times N$ onto G . (E.g., for $\mathfrak{sl}(n, \mathbf{R})$, \mathfrak{k} is the rotation algebra, \mathfrak{a} is the algebra of diagonal matrices with null trace, \mathfrak{n} is the algebra of strictly triangular matrices; at the group level this decomposition can be naturally obtained through the Gram-Schmidt orthogonalization process.)

To the root decomposition (51) of \mathfrak{g}_C there corresponds a restricted root decomposition

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{m} + \sum_{\alpha' \in \mathcal{A}'} \mathfrak{g}^{\alpha'}$$

where \mathfrak{m} are the elements of \mathfrak{k} which commute with all the elements of \mathfrak{a} , \mathcal{A}' is the set of nonzero restrictions α' of α to \mathfrak{a} and

$$\mathfrak{g}^{\alpha'} = \{X \in \mathfrak{g} : \text{ad}(H) X = \alpha'(H) X \text{ for all } H \in \mathfrak{a}\}$$

Since different roots α can have the same restriction α' to \mathfrak{a} , the multiplicity

$$m_{\alpha'} = \dim \mathfrak{g}^{\alpha'} \quad (59)$$

can be higher than 1 for some algebras. The Cartan involution changes the sign of the restricted roots:

$$\theta \mathfrak{g}^{\alpha'} = \mathfrak{g}^{-\alpha'} \quad (60)$$

The algebra \mathfrak{k} can therefore be decomposed as

$$\mathfrak{k} = \mathfrak{m} + \sum_{\alpha' \in \mathcal{P}'} (\mathfrak{g}^{\alpha'} + \theta \mathfrak{g}^{\alpha'}) \quad (61)$$

A Weyl basis $\{X_\alpha\}$ for \mathfrak{g}_C can be projected on \mathfrak{g} by

$$Y_\alpha = \frac{1}{2} (X_\alpha + \sigma X_\alpha) \quad (62)$$

so that $\mathfrak{g}^{\alpha'} = \sum \mathbf{R} Y_\alpha$ where the sum runs over all $\alpha \in \mathcal{A}$ whose restriction to \mathfrak{a} is α' .

On a Lie group one can define the (right invariant) Haar measure

$$dg = \bigwedge_{l=1}^{\dim G} g^{-1} \delta_l g$$

where $\delta_l g = g \delta_l X$ and the $\delta_l X$ are a basis of 1-forms on \mathfrak{g} . Semisimple Lie groups are unimodular, i.e., such that $|\det \text{Ad}(g)| = 1$ for each $g \in G$, so that right invariant measures are also left invariant. The Iwasawa decomposition induces the decomposition

$$dg = j(k, a, n) dk da dn \quad (63)$$

where j is a Jacobian factor. The groups G, K, A, N are all unimodular, therefore both sides of (63) are invariant under the right and left translations $g \rightarrow k_R g, g \rightarrow g n_L$ with $k_R \in K$ and $n_L \in N$, so that $j(k, a, n) = j(a)$. On the other hand, the l.h.s. of (63) is also invariant under $g \rightarrow ga^{-1}$, which corresponds to $kan \rightarrow ke \cdot (ana^{-1})$ (with e the identity element), so that

$$\begin{aligned} j(a) &= j(e) \frac{\partial(\text{ana}^{-1})}{\partial(n)} \\ &= j(e) \det(\text{Ad}(a)|_n) \\ &= j(e) \exp[\text{Tr ad}(\exp^{-1} a)|_n] \\ &= j(e) \exp \left[\sum_{\alpha \in P} \alpha(\exp^{-1} a) \right] \end{aligned}$$

If the measures on G, K, A, N are conveniently normalized we can assume that $j(e) = 1$, so

$$dg = \exp[2\rho(\exp^{-1} a)] dk da dn \quad (64)$$

with

$$\rho = \frac{1}{2} \sum_{\alpha \in P} \alpha \quad (65)$$

Let τ be a skew-hermitian representation of the compact form \mathfrak{u} on a finite dimensional Hilbert space V . By analytically extending τ to the complexification $\mathfrak{g}_C = \mathfrak{u} + i\mathfrak{u}$ and then restricting to the real form $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ one obtains a representation of the real algebra \mathfrak{g} (Weyl unitary trick). If $K \in \mathfrak{k}, P \in \mathfrak{p}$, then $\tau(K)$ and $\tau(iP)$ are skew-hermitian, while $\tau(P)$ is hermitian. In other words,

$$\tau^\dagger(X) = -\tau(\theta X)$$

Denoting $T(\exp(X)) = \exp(\tau(X))$, this property exponentiates (ref. 27, p. 144) to

$$T^\dagger(g) = T^{-1}(\theta g) \quad (66)$$

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