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The Lyapunov Spectrum of a Continuous Product of Random Matrices

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Abstract

We expose a functional integration method for the averaging of continuous products $\hat{P}_t$ of $N \times N$ random matrices. As an application, we compute exactly the statistics of the Lyapunov spectrum of $\hat{P}_t$. This problem is relevant to the study of the statistical properties of various disordered physical systems, and specifically to the computation of the multipoint correlators of a passive scalar advected by a random velocity field. Apart from these applications, our method provides a general setting for computing statistical properties of linear evolutionary systems subjected to a white noise force field.

Key words: Lyapunov exponents, random matrices, functional integral, disordered systems, passive scalar, Gauss decomposition, loop groups.

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1 Introduction

In this work we give a detailed exposition of a functional integral method for the averaging of time-ordered exponentials of $N \times N$ random matrices which has found several applications in the study of the statistical properties of disordered systems.

The method was introduced by one of the authors [2] in the $N = 2$ case in order to compute the partition function of the Heisenberg ferromagnet, and was thereafter applied to the study of one-dimensional Anderson localization [3] and to some problems of mesoscopic physics [4]. Later, the same technique [6, 7] allowed to obtain analytical results in the problem of a passive scalar advected by a 2-dimensional random velocity field. The approach of refs. [6, 7] was then extended to the more general N-dimensional case in ref. [1].

In the present work we expose the method in its full generality and show that it allows to compute exactly the statistics of the whole Lyapunov spectrum of the matrix $\hat{P}_t$ describing the time evolution of a linear system subjected to a white noise force field. Such a spectrum is relevant [8] in the computation of the multipoint correlators of the passive scalar (as well as in the computation of the correlators of passive vectors and tensors); in this case the matrix $\hat{P}_t$ describes the time evolution of particles in a turbulent fluid, linearized around a given trajectory.

However, our setting presents a high degree of generality and therefore a wider range of applications. In particular, our formalism can be naturally applied to the study of N-level quantum-mechanical systems affected by a random noise, and it is complementary to the supersymmetric approach [14, 15] to the problem of $N/2$ channels localization in a disordered conductor.

2 Averages of time-ordered exponentials

Let us start with the problem of computing gaussian averages of the form

$$\left\langle \mathcal{F}[\hat{P}_t] \right\rangle = \frac{1}{\mathcal{N}} \int \mathcal{D}\hat{X} \exp\left(-S[\hat{X}]\right) \mathcal{F}[\hat{P}_t] \tag{1}$$

where $\hat{X}(s)$, for $0 \leq s \leq T$, is a traceless $N \times N$ hermitian matrix,

$$\mathcal{D}\hat{X} \equiv \prod_{0\leq s\leq T} \prod_{i<j} dX_{ij}(s) dX_{ji}(s) \prod_i dX_{ii}(s) \tag{2}$$

is the Feynman-Kac measure, $\mathcal{N}$ is chosen in such a way that $\langle 1 \rangle = 1$,

$$S[\hat{X}] = \frac{1}{4D} \int_0^T \mathrm{Tr}\, \hat{X}^2(s)\, ds \tag{3}$$

and $\hat{P}_t$ is the time-ordered exponential

$$\hat{P}_t = \mathcal{T} \exp\left(\int_0^t \hat{X}(s)\, ds \right) \tag{4}$$

such that $\mathbf{r}(t) = \hat{P}_t \mathbf{r}_0$ is the general solution of the linear problem $\dot{\mathbf{r}} = \hat{X}\mathbf{r}$, $\mathbf{r}(0) = \mathbf{r}_0$ and

$$\dot{\hat{P}}_t \hat{P}_t^{-1} = \hat{X} \tag{5}$$

We shall now introduce a set of "collective" integration variables in order to re-express (4) in a more tractable form. At the same time, we shall chose the new variables in such a way that (3) still be quadratic and that the Jacobian determinant of the functional transformation be particularly simple.

As a first step, let us Gauss-decompose the matrix $\hat{P}_t$:

$$\hat{P}_t = (\mathbf{1} + \hat{\phi}(t)) \cdot \exp(\hat{\tau}(t)) \cdot (\mathbf{1} + \hat{\theta}(t)) \tag{6}$$

where

$$\begin{aligned} \phi_{ij}(t) &\equiv 0, \qquad i \leq j \\ \theta_{ij}(t) &\equiv 0, \qquad i \geq j \\ \tau_{ij}(t) &\equiv \tau_i(t)\,\delta_{ij} \end{aligned} \tag{7}$$

$$\tau_N(t) \equiv -\sum_{j=1}^{N-1} \tau_j(t) \tag{8}$$

Moreover, in order to ensure the equality $\hat{P}_0 = \mathbf{1}$ we shall impose

$$\hat{\phi}(0) = \mathbf{0}, \qquad \hat{\theta}(0) = \mathbf{0} \tag{9}$$

We would like now to re-express the "local" degrees of freedom $X_{ij}(t)$ in terms of the global ones $\phi_{ij}(t), \tau_{ij}(t), \theta_{ij}(t)$. This can be accomplished by making use of the basis $\hat{e}_{ij}$ of the matrix algebra, which is defined by $(\hat{e}_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and satisfies the commutation rules

$$[\hat{e}_{ij}, \hat{e}_{kl}] = \delta_{jk}\hat{e}_{il} - \delta_{il}\hat{e}_{kj} \tag{10}$$

In particular, one has

$$\begin{aligned} \hat{e}_{ii}\hat{e}_{kl} &= \hat{e}_{kl}(\hat{e}_{ii} + \delta_{ik} - \delta_{il}) \\ e^{\hat{\tau}}\,\hat{e}_{ij}\,e^{-\hat{\tau}} &= e^{\tau_i - \tau_j}\,\hat{e}_{ij} \end{aligned} \tag{11}$$

From these relations the desired expression for X_{ij} readily follows:

$$\begin{aligned} X_{ij} = {} & \dot{\phi}_{ij} + \sum_k \dot{\phi}_{ik}\tilde{\phi}_{kj} \\ & + \dot{\tau}_i\delta_{ij} + \phi_{ij}\dot{\tau}_j + \dot{\tau}_i\tilde{\phi}_{ij} + \sum_k \phi_{ik}\dot{\tau}_k\tilde{\phi}_{kj} \\ & + A_{ij} + \sum_k(\phi_{ik}A_{kj} + A_{ik}\tilde{\phi}_{kj}) + \sum_{k,l}\phi_{ik}A_{kl}\tilde{\phi}_{lj} \end{aligned} \tag{12}$$

where $A_{ij} \equiv e^{\tau_i - \tau_j} \sum_k \dot{\theta}_{ik}(\delta_{kj} + \tilde{\theta}_{kj})$,

$$\tilde{\phi}_{ij} \equiv -\phi_{ij} + \sum_k \phi_{ik}\phi_{kj} - \sum_{k,l}\phi_{ik}\phi_{kl}\phi_{lj} + \cdots \tag{13}$$

and a similar definition holds for $\tilde{\theta}_{ij}$ (for any fixed N (13) is a finite sum, since $\hat{\phi}$ is a nilpotent matrix; the same is true for $\hat{\theta}$).

Substituting (12) in (3) one obtains

$$\frac{1}{2}\,\mathrm{Tr}\,\hat{X}^2 = \frac{1}{2}\sum_{j=1}^{N}\dot{\tau}_j^2 + \sum_{i,j}\left(\dot{\phi}_{ij} + \sum_{k=1}^{N}\tilde{\phi}_{ik}\dot{\phi}_{kj}\right)e^{\tau_j-\tau_i}\left(\sum_{k=1}^{N}\dot{\theta}_{jl}\tilde{\theta}_{li} + \dot{\theta}_{ji}\right) \tag{14}$$

The form of (14) suggests the introduction of the new variables

$$\begin{aligned}\bar{\phi}_{ij} &= \sum_{k,l}\dot{\theta}_{ik}\left(\delta_{kl} + \tilde{\theta}_{kl}\right)\left(\delta_{lj} + \tilde{\phi}_{lj}e^{\tau_j-\tau_l}\right)e^{\tau_i-\tau_j}, \qquad i<j \\ \bar{\phi}_{ij} &\equiv 0, \qquad i \geq j\end{aligned} \tag{15}$$

so that

$$\frac{1}{2}\,\mathrm{Tr}\,\hat{X}^2 = \frac{1}{2}\sum_{j=1}^{N}\dot{\tau}_j^2 + \sum_{i,j}\dot{\phi}_{ij}\bar{\phi}_{ji} \tag{16}$$

Relation (15) can be inverted giving

$$\dot{\theta}_{ij} = \sum_{k,l}\bar{\phi}_{ik}e^{\tau_k-\tau_i}\left(\delta_{kl} + \phi_{kl}e^{\tau_l-\tau_k}\right)\left(\delta_{lj} + \theta_{lj}\right)\chi_{li} \tag{17}$$

where

$$\chi_{ij} \equiv 1 - \bar{\chi}_{ij} \equiv \begin{cases} 1, & i>j \\ 0, & i \leq j \end{cases} \tag{18}$$

Through (17) one can re-express the θ_{ij} as functions of the new variables ϕ_{ij}, τ_i and $\bar{\phi}_{ij}$ in a recursive way, thanks to the "triangular" form of the equation. For instance, for $N=3$ one gets

$$\begin{aligned}\theta_{23}(t) &= \int_0^t \bar{\phi}_{23}(s)e^{-\tau_1(s)-2\tau_2(s)}\,ds, \\ \theta_{12}(t) &= \int_0^t A(s)\,ds, \qquad \text{where} \quad A = (\bar{\phi}_{12} + \bar{\phi}_{13}\phi_{32})e^{\tau_2-\tau_1}\,, \\ \theta_{13}(t) &= \int_0^t [A(s)\theta_{23}(s) + \bar{\phi}_{13}(s)e^{-2\tau_1(s)-\tau_2(s)}]\,ds\end{aligned} \tag{19}$$

As a matter of fact, for any fixed i the $N-i$ functions θ_{ij} can be expressed through the $N-i-1$ functions $\theta_{i+1,j}$ and the remaining variables by means of a single quadrature. This is an important point, since for practical calculations $\hat{P}_t$ has to be re-expressed in terms of the new variables $\hat{\phi}, \dot{\hat{\tau}}, \hat{\bar{\phi}}$.

We must now substitute $\hat{X}(s)$ as an integration variable in the functional integral (1) with the new variables $\hat{\phi}(s), \dot{\hat{\tau}}(s), \hat{\bar{\phi}}(s)$. Again using the commutation rules (10), and renaming $\dot{\tau}_i \equiv \rho_i$ for convenience, we finally get

$$\begin{aligned}X_{ij} &= \bar{\phi}_{ij} + \sum_k \phi_{ik}\bar{\phi}_{kj}, \qquad i<j \\ X_{ii} &= \rho_i + \sum_k(\phi_{ik}\bar{\phi}_{ki} - \bar{\phi}_{ik}\phi_{ki}), \qquad i = 1,\ldots,N\end{aligned} \tag{20}$$

$$\begin{aligned} X_{ij} &= \phi_{ij}\rho_j + \rho_i\tilde{\phi}_{ij} + \dot{\phi}_{ij} \\ &+ \sum_k (\phi_{ik}\rho_k\tilde{\phi}_{kj} + \dot{\phi}_{ik}\tilde{\phi}_{kj} + \phi_{ik}\bar{\phi}_{kj} - \bar{\phi}_{ik}\phi_{kj}) \\ &- \sum_{k,l} (\bar{\chi}_{jk}\phi_{ik}\bar{\phi}_{kl}\phi_{lj} + \bar{\chi}_{li}\bar{\phi}_{ik}\phi_{kl}\tilde{\phi}_{lj}) \\ &- \sum_{k,l,m} \bar{\chi}_{mk}\phi_{ik}\bar{\phi}_{kl}\phi_{lm}\tilde{\phi}_{mj}, \qquad i > j \end{aligned} \tag{21}$$

In ref. [1] the $N = 3$ case was explicitly considered. We observe that the matrix elements of $\hat{X}(t) = \dot{\hat{P}}_t \hat{P}_t^{-1}$ transform as

$$X_{ij}(t) \to e^{\sigma_{ij}} X_{ij}(t) \tag{22}$$

under the global gauge transformation

$$\phi_{ij}(t) \to e^{\sigma_{ij}} \phi_{ij}(t), \qquad \bar{\phi}_{ij}(t) \to e^{\sigma_{ij}} \bar{\phi}_{ij}(t), \qquad \rho_i(t) \to \rho_i(t) \tag{23}$$

where σ_{ij} satisfies $\sigma_{ik} + \sigma_{kj} = \sigma_{ij}$, $\sigma_{ij} = -\sigma_{ji}$.

As the last step, we must compute the Jacobian determinant of the functional transformation (21). Notice first that through the shift given by

$$\begin{aligned} \bar{\phi}_{ij} &\to \bar{\phi}_{ij} - \sum_{k=1}^{N} \phi_{ik}\bar{\phi}_{kj} \\ \rho_i &\to \rho_i - \sum_{k=1}^{N} (\phi_{ik}\bar{\phi}_{ki} - \bar{\phi}_{ik}\phi_{ki}) \end{aligned} \tag{24}$$

(which has Jacobian $\mathcal{J}' = 1$) one can reduce to the computation of

$$\mathrm{Det}\left(\frac{\delta \hat{X}_-}{\delta\hat{\phi}\,\delta\hat{\rho}\,\delta\hat{\bar{\phi}}}\right) \tag{25}$$

where $\hat{X}_-$ is the strictly lower triangular part of $\hat{X}$. The Jacobian (25) can be computed by means of the standard regularization procedure (see ref. [2])

$$\begin{aligned} &\hat{\phi}_n = \hat{\phi}(t_n), \qquad \hat{\rho}_n = \hat{\rho}(t_n), \qquad \hat{\bar{\phi}}_n = \hat{\bar{\phi}}(t_n) \\ &t_n = hn, \quad n = 1, \ldots, M, \quad h = T/M \to 0, \quad M \to +\infty \\ &\dot{\hat{\phi}}(t) \to \frac{\hat{\phi}_n - \hat{\phi}_{n-1}}{h}, \qquad \hat{\phi}(t) \to \frac{\hat{\phi}_n + \hat{\phi}_{n-1}}{2} \end{aligned} \tag{26}$$

giving

$$\mathcal{J} \propto \exp\left(\sum_{j=1}^{N-1} (N-j) \int_0^T \rho_j(s)\, ds\right) \tag{27}$$

Applying now the variable transformation $\hat{X} \to (\hat{\phi}, \hat{\rho}, \hat{\bar{\phi}})$ one sees that the functional integral (1) reduces to

$$\left\langle \mathcal{F}[\hat{P}_t] \right\rangle = \frac{1}{\mathcal{N}'} \int \mathcal{D}\hat{\phi}\,\mathcal{D}\hat{\rho}\,\mathcal{D}\hat{\bar{\phi}}\, \exp\left(-S'[\hat{\phi}, \hat{\rho}, \hat{\bar{\phi}}]\right) \mathcal{F}[(\mathbf{1} + \hat{\phi})e^{\hat{\tau}}(\mathbf{1} + \hat{\theta})] \tag{28}$$

where $\hat{\theta} = \hat{\theta}[\hat{\phi}, \hat{\rho}, \hat{\bar{\phi}}]$ is obtained by solving (17), $\hat{\tau} = \int_0^T \hat{\rho}(s)\, ds$, the ρ_i are constrained by (8), $\mathcal{N}'$ is the normalization factor and

$$S' = \frac{1}{2D} \int_0^T \left(\frac{1}{2} \sum_{k=1}^{N} \rho_k^2 + \sum_{i,j} \dot{\phi}_{ij} \bar{\phi}_{ji} - 2D \sum_{k=1}^{N-1} (N-k) \rho_k \right) ds \tag{29}$$

In (1) the functional integration is constrained to the surface

$$\Gamma_0 = \{ X_{ij}(s) = X_{ji}^*(s), \quad 0 \leq s \leq T \} \tag{30}$$

In refs. [2, 7] it was shown, using the Cauchy theorem, that whenever $\mathcal{F}$ is holomorphic in the matrix elements X_{ij} one can modify the integration surface Γ_0 to the homotopically equivalent

$$\Gamma_1 = \{ \phi_{ij}(s) = \bar{\phi}_{ji}^*(s),\ \mathrm{Im}\, \rho_i(s) = 0, \quad 0 \leq s \leq T \} \tag{31}$$

without affecting the value of the integral. This means that in (28) $\hat{\bar{\phi}}$ may be regarded as the hermitian conjugate of $\hat{\phi}$.

We would like to remark that expressions similar to (21) were obtained in the framework of conformal field theory [16]. However, the explicit form of the variables $\bar{\phi}$ and of the Jacobian $\mathcal{J}$, which are essential for any physical application of our method, were not computed there.

3 The Lyapunov Spectrum

We shall now define the Lyapunov exponents λ_j, $j = 1, \ldots, N$, according to the relation [9]

$$\lambda_1 + \cdots + \lambda_k = \frac{1}{T} \log \mathrm{Vol}(\hat{P}_T \mathbf{v}_1, \ldots, \hat{P}_T \mathbf{v}_k) \tag{32}$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is an orthonormal set of vectors generating a unitary k-volume. For the sake of definiteness we shall choose $\mathbf{v}_j = \mathbf{e}_j$, where $(\mathbf{e}_j)_i = \delta_{ij}$. One has

$$\mathrm{Vol}(\hat{P}_T \mathbf{e}_1, \ldots, \hat{P}_T \mathbf{e}_k) = \sqrt{\sum_{a=1}^{l_k} \Delta_a^2(\hat{M}_k)} \tag{33}$$

where $\Delta_a(\hat{M}_k)$, $a = 1, \ldots, l_k$, $l_k \equiv \binom{n}{k}$ are the $k \times k$ minors of the $n \times k$ matrix $\hat{M}_k = [\hat{P}_T \mathbf{e}_1, \ldots, \hat{P}_T \mathbf{e}_k]$. Let $\vec{\phi}_j, \mathbf{p}_j$, $j = 1, \ldots, N$ be the vectors defined by $(\vec{\phi}_j)_i = \delta_{ij} + \phi_{ij}$, $(\mathbf{p}_j)_i = (\hat{P}_T)_{ij}$. Then

$$\begin{aligned} \mathbf{p}_j &= \sum_{i,k} (\delta_{ik} + \phi_{ik})\, e^{\tau_k} (\delta_{kj} + \theta_{kj})\, \mathbf{e}_i \\ &= \sum_{k=1}^{N} e^{\tau_k} (\delta_{kj} + \theta_{kj})\, \vec{\phi}_k \\ &= e_j^{\tau} \vec{\phi}_j + \sum_{k<j} \theta_{kj} e^{\tau_k} \vec{\phi}_k \end{aligned} \tag{34}$$

From (34) and the multilinearity of determinants it follows

$$\begin{aligned}
\Delta_a(M_k) &= \Delta_a[\mathbf{p}_1, \ldots, \mathbf{p}_n] \\
&= \Delta_a[e^{\tau_1}\vec{\phi}_1, \ldots, e^{\tau_k}\vec{\phi}_k] \\
&= e^{\tau_1+\cdots+\tau_l}\left(\delta_{a,1} + \sum_{r_{ij}\geq 1} c_{ij}^{a,k}\phi_{ij}^{r_{ij}}\right)
\end{aligned} \tag{35}$$

where Δ_1 is the minor obtained from the first k rows of M_k, r_{ij} are strictly positive integers, and $c_{ij}^{a,k}$ are integer coefficients with $c_{ij}^{1,k} \equiv 0$. One has then

$$\begin{aligned}
\lambda_1 + \cdots + \lambda_k &= \frac{1}{T}[\tau_1(T) + \cdots + \tau_k(T))] \\
&+ \frac{1}{2T}\log\left[1 + \sum_{a=2}^{l_k}\left(\sum_{r_{ij}\geq 1} c_{ij}^{a,k}\phi_{ij}^{r_{ij}}(T)\right)^2\right]
\end{aligned} \tag{36}$$

and

$$\lambda_k = \frac{1}{T}\int_0^T \rho_k\, dt + \frac{1}{2T}[\log(1 + f_k(\hat{\phi})) - \log(1 + f_{k-1}(\hat{\phi}))] \tag{37}$$

where $1 + f_k(\hat{\phi})$ is the argument of the logarithm in (36).

Let us now compute the probability distribution function for λ_k. The form of (29) implies that the ϕ-dependent terms in (37) give no contribution, since they do not contain the conjugate variables $\bar{\phi}_{ij}$.

We are therefore left with $N-1$ Gaussian integrations over $\rho_1, \ldots, \rho_{k-1}, \rho_{k+1}, \ldots, \rho_N$ which give the following exact result for the statistics of ρ_k:

$$\mathcal{D}\rho_k \exp\left(-\frac{N}{4D(N-1)}\int_0^T (\rho_k(s) - \bar{\lambda}_k)^2\, ds\right) \tag{38}$$

where

$$\bar{\lambda}_k = D(N - 2k + 1), \qquad k = 1, \ldots, N \tag{39}$$

The probability distribution function $p(\lambda_k; T)$ of the k-th Lyapunov exponent λ_k is then:

$$p(\lambda_k; T) = \frac{1}{2}\sqrt{\frac{NT}{\pi D(N-1)}}\exp\left(-\frac{NT}{4D(N-1)}(\lambda_k - \bar{\lambda}_k)^2\right) \tag{40}$$

The Lyapunov exponents λ_k are statistically dependent due to the constraint (8) and their joint distribution function has a generalized Gaussian form [3]. We have thus obtained a complete knowledge of the statistics of the Lyapunov spectrum of the matrix $\hat{P}_t$. This has an essential application to the problem of the computation of the multipoint correlators of a passive scalar advected by a random velocity field (see the Appendix).

[3]The Gaussian distribution of the Lyapunov exponent in the $N = 2$ case was obtained in the context of the passive scalar problem in ref. [6].

4 Conclusions

In this work we gave a detailed exposition in the general $N \times N$ case of a functional integral method for the averaging of time-ordered exponentials of random matrices which has found several applications to the study of the statistical properties of disordered physical systems [2, 3, 4, 1, 6, 7, 8] and we have shown how the statistics of the Lyapunov spectrum of a linear evolutionary process can be computed exactly. As a matter of fact, our method provides a general setting for computing the statistics of linear evolutionary systems subjected to a white noise force field.

We would like to conclude with some remarks. The definition of the Lyapunov exponents as the logarithmic rate of growth of a k-dimensional parallelepiped (see eq. (32) and ref. [9]) is the most natural from a physical point of view, e.g. in the passive scalar problem. Generally speaking these exponents do not coincide with the logarithms of the eigenvalues of the evolution matrix $\hat{P}_t$. The statistics of the eigenvalues of a similar evolution matrix was studied in refs. [10, 11, 12, 13]. Our method, however, allows one to obtain a more detailed statistical information about the evolution of initial vectors and to compute non-trivial correlation functions of their components. For an application to the passive scalar problem see ref. [8].

Lastly, we would like to remark that a more refined application of the functional integral method we described allowed to solve effectively the more difficult case of a “coloured” noise (see ref. [7]).

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Appendix

The method we exposed in this work has a direct application to the computation of the statistics of a scalar passively advected by a random velocity field. In order to illustrate this point we will briefly recall here the terms of the problem. For more detail see refs. [5, 6, 7, 1, 8].

The evolution of a scalar field $\theta(\mathbf{r}, t)$ passively advected by a velocity field $\mathbf{v}(\mathbf{r}, t)$ and generated by a source $\phi(\mathbf{r}, t)$ is given by

$$\dot{\theta} + \mathbf{v} \cdot \nabla \theta = \phi \tag{41}$$

If we impose on (41) the asymptotic condition $\theta(\mathbf{r}, -\infty) = 0$ we get the solution

$$\theta(\mathbf{r}, t) = \int_0^{+\infty} \phi(\mathbf{R}(\mathbf{r}, t-s), t-s)\, ds \tag{42}$$

saying that $\theta(\mathbf{r}, t)$ is completely determined in terms of the trajectories $\mathbf{R}(\mathbf{r}_0, t)$ of the fluid particles:

$$\dot{\mathbf{R}} = \mathbf{v}(\mathbf{R}, t), \qquad \mathbf{R}(\mathbf{r}_0, 0) = \mathbf{r}_0 \tag{43}$$

Let us now take ϕ and $\mathbf{v}$ to be random, δ-correlated in time fields . The source ϕ will be assumed to be spatially correlated on a scale L. The velocity field $\mathbf{v}$ might be multi-scale, with smallest scale larger or of the order of L. The statistics of ϕ and $\mathbf{v}$ will be assumed to be spatially isotropic.

Generally speaking, one is interested in computing equal-time correlators of the form $\langle \theta(\mathbf{r}_1, 0)\theta(\mathbf{r}_2, 0)\rangle$ for $|\mathbf{r}_2 - \mathbf{r}_1| \ll L$. From the isotropicity of the statistics of ϕ and $\mathbf{v}$ it follows that such quantities are rotation-invariant. Moreover, (42) implies that the statistics of θ is completely determined in terms of the statistics of the trajectories (43).

In order to subtract the effect or sweeping, let us chose a reference frame locally comoving with one of the fluid particles (see refs. [5, 6, 7]). We can then locally linearize (43), obtaining

$$\dot{\mathbf{R}} \simeq \hat{\sigma}(t)\mathbf{R} \tag{44}$$

where $\sigma_{ij} \equiv \partial v_j / \partial r_i$, the matrix of velocity derivatives, will be taken to be a random gaussian process. In the general case we have $\hat{\sigma} = \hat{R} + \hat{S}$, where $\hat{R}$ is the antisymmetric part of $\hat{\sigma}$, inducing a rotation of the passive scalar blob, and $\hat{S}$ is the symmetric part, representing the stretching of the unit blob. We will consider the case of an incompressible fluid, so $\mathrm{Tr}\,\hat{S} = 0$.

More specifically, let us consider the following statistics of $\hat{\sigma}$:

$$\mathcal{DM}[\hat{\sigma}] = \mathcal{D}\hat{\sigma} \exp\left(-\frac{1}{2D_s}\int_0^T \mathcal{L}\, dt\right), \tag{45}$$

$$\mathcal{L} = \frac{1}{2}\left(\mathrm{Tr}\,\hat{S}^2 - \frac{D_s}{D_r}\mathrm{Tr}\,\hat{R}^2\right) = \frac{1}{2}\mathrm{Tr}\,\left(\hat{S} + i\sqrt{\frac{D_s}{D_r}}\hat{R}\right)^2$$

Since we are interested in rotation-invariant quantities the final result shall be independent on D_r. This arbitrariness allows one to substitute $\sqrt{D_s/D_r} \to -i$, $\hat{R} \to i\hat{R}$, $\hat{\sigma} \to \hat{X} = \hat{S} + i\hat{R}$ (see ref. [1]), and thus to consider a generic traceless hermitian matrix $\hat{X}$ with averaging weight $\exp\left(-\frac{1}{2D_s}\int \frac{1}{2}\mathrm{Tr}\,\hat{X}^2\right)$ in the place of the generic traceless real matrix $\hat{\sigma}$ with the averaging weight (45): this allows one to refer to the results of sect. 2.

From rotational invariance it follows that the statistics of θ is completely determined in terms of the statistics of the Lyapunov spectrum of the matrix $\hat{P}_t$ defined by

$$\dot{\hat{P}}_t = \hat{X}\hat{P}_t, \qquad \hat{P}_0 = \mathbf{1} \tag{46}$$

This reduces us to the problem studied in the preceding sections. The Gaussian statistics of the Lyapunov exponents agrees with an old result [17] about the Gaussian statistics of a line element in a δ-correlated in time velocity field.

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