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Calculation of the tensile and flexural strength of disordered materials using fractional calculus

Alberto Carpinteri [a,*], Pietro Cornetti [a], Kiran M. Kolwankar [b,c,1]

[a] Department of Structural and Geotechnical Engineering, Politecnico di Torino, 10129 Torino, Italy
[b] Centre de Mathématiques et de leurs Applications, Ecole Normale Supérieure, CNRS, 94140 Cachan, France
[c] Laboratoire de Physique de la Matière Condensée, Ecole Polytechnique, CNRS, 91128 Palaiseau, France

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Abstract

Aim of the present paper is to show an application of the fractal integral, which was recently introduced in the framework of fractional calculus, to the mechanics of materials with disordered microstructure. Some rules of integration for simple functions performed on generalized Cantor sets have been developed. We believe that these results can be of some interest in the modeling of materials whose microstructure is fractal-like. Part of the paper is devoted, as an example, to show why concrete can be considered such a material. The fractal dimension of the stress carrying material ligament is found. Finally, by the fractal integration rules previously developed, the computation of the tensile and flexural strength for structures made by a concrete-like material is performed and consequent size effects are highlighted.

1. Introduction

One of the main goals of solid mechanics during the last two decades has been the understanding of the so-called size effects. Generally, with size effect we mean a different structural behavior varying the structural size. For instance, experiments have shown that the strength of several materials is not an exclusive property of the material since it is also a function of the size of the structure made by that material.

Size effects are particularly relevant in materials with disordered microstructures, like concrete and rocks. Since these materials have a brittle behavior for what concerns the bearing load but, at the same time, can dissipate a significant amount of energy after reaching the peak load (i.e. during the softening regime), they are also referred to as quasi-brittle materials. Several Authors think that the reason of the rising of size effects lies in the heterogeneity of the material microstructure. This explains the importance given to microstructure and multi-scale models in the recent years.

Disorder seems to be the main characteristic of quasi-brittle material microstructure. In an attempt to model the disorder, it has been pointed out that the irregularities of the microstructure look the same at any resolution scale. This property is called self-similarity and has been detected, for instance, in concrete crack surfaces [1], as well as in rock porosity [2] or shear bands [3]. In general, the porosity or the disorder or the non-linearity in the governing constitutive equations lead to the concentration of the stress on sets that are self-similar over a broad range of length scales. Therefore, a mathematical fractal set can be used to approximate the system between the lower and upper cut off length scales beyond which the system fails to be self-similar. This means that, even if difficult to handle, disorder can be

* Corresponding author. Tel.: +39-11-5644866; fax: +39-11-5644899.
E-mail address: carpinteri@polito.it (A. Carpinteri).
[1] Now at: Max Planck Institute for Mathematics in the Sciences, D – 04103, Leipzig, Germany.

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described satisfactorily by means of non-classical geometrical sets such as the fractal sets [4]. In the context of fracture of disordered media, the fractal approximations have been successfully used (see for example [5]). Numerical simulations have also been carried out to study the distribution of stress on the surface of a fractal embedded in an elastic medium [6]. Indeed, fractal geometry is one of the most powerful tools to deal with structures presenting the same non-smooth aspect at any resolution level [7]. Even more interesting, in fractal geometry, the sets are described mainly by their own dimension. Fractal dimension implies existence of correlations over many scales. Therefore, it is the parameter that influences the global behavior of the macrostructure and the related size effects. The knowledge of the fractal dimension (or of the fractal dimensions, if there are more than one) characterizing the material microstructure is therefore of maximum importance. On the other hand, the fractal dimension alone does not completely characterize the given set since two different sets with the same fractal dimension can have different lacunarity.

The general question we wonder is whether it is possible to generalize the formulation of classical continuum mechanics to naturally include self-similar structures. Unfortunately, functions encountered in fractal analysis are irregular and do not possess even first order derivative at the points of interest. This means that the usual calculus is inadequate to handle fractal structures and processes. Therefore, it is argued that a new calculus should be developed which includes fractal structures intrinsically. Some Researchers wondered whether Fractional Calculus, i.e. the branch of the calculus dealing with integrals and derivatives of arbitrary order, could be such a candidate [8]. There have been some attempts in this direction using either just fractional calculus [9,10] or some other techniques [11].

More recently, new mathematical operators, viz, the local fractional derivative and the fractal integral, were introduced in [12]. These operators, which are based on usual fractional calculus, appear to be useful in the description of fractal structures and processes. It is important to emphasize that, what seems to be really interesting in studying fractals via fractional calculus, are the non-integer physical dimensions that arise dealing with both fractional operators and fractal sets.

These developments give a possible direction along which one can proceed for generalizing classical continuum mechanics [13]. In addition to the proper calculus, one will, of course, need to modify some concepts and introduce new definitions which will make sense in fractal spaces. At present, we take a formal approach and make simple calculations. We postpone issues like mathematical rigor and making connections to the governing physical equations. As a first step, we restrict our calculations to one dimensional cases. This paper follows a previous one [14] in which the fractional operators introduced in [15,16] have been applied to develop a model able to describe the stress and strain (fractal) localization in materials with disordered microstructure under tensile loads. Aim of the present paper is to show a simple but consistent application of the fractal integrals to the computation of the tensile and flexural strength of this kind of materials. The plan of the paper is as follows. In the next section we start by reviewing the definitions of the classical fractional integral and of the fractal integral. Then, we devote Section 3 to the development of some rules for (fractal) integration performed on generalized Cantor sets with dimension comprised between 0 and 1. Section 4 shows, as an example, why concrete should be modeled by a fractal microstructure and explains how to compute its dimension. Eventually, in Section 5, we apply the integration rules previously developed to the computation of the strength of structures made by concrete-like materials.

2. The fractional calculus and the fractal integral

While classical calculus treats integrals and derivatives of integer order, fractional calculus is a branch of mathematics that deals with the generalization of integrals and derivatives to all real (and even complex) orders. There are various definitions of fractional differintegral operators, not necessarily equivalent to each other. A complete list of these definitions can be found in the fractional calculus treatises [17–20]. These definitions have different origins. The most frequently used definition of a fractional integral of order α ($\alpha > 0$) is the Riemann–Liouville definition, which is a straightforward generalization to non-integer values of Cauchy formula for repeated integration:

$$\frac{\mathrm{d}^{-\alpha}f(x)}{[\mathrm{d}(x-a)]^{\alpha}} = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(y)}{(x-y)^{1-\alpha}}\mathrm{d}y \tag{1}$$

where we used the notation by Oldham and Spanier [17]; a is the lower limit of integration while $\Gamma(x)$ is the Gamma function. These operators and the equations involving them have found many applications during the last decade [21,22]. Similarly, one can also define a derivative of fractional order (between 0 and 1) by

$$\frac{\mathrm{d}^{\alpha}f(x)}{[\mathrm{d}(x-a)]^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{f(y)}{(x-y)^{\alpha}}\mathrm{d}y \tag{2}$$

Looking for a link between fractional calculus and fractals, it is worthwhile to cite the scaling property (for $a = 0$):

$$\frac{\mathrm{d}^{-\alpha} f(bx)}{[\mathrm{d}(bx)]^{-\alpha}} = b^{\alpha} \frac{\mathrm{d}^{-\alpha} f(bx)}{[\mathrm{d}x]^{-\alpha}} \tag{3}$$

It means that the fractional integral operator shows the same scaling laws as the α-dimensional Hausdorff measure of a fractal set $V : \mathscr{M}_{\alpha}(bV) = b^{\alpha}\mathscr{M}_{\alpha}(V)$. For the scaling property (3) in the case $a \neq 0$, see [17].

Recently, a local definition of the fractional derivative [12,23] was introduced. According to this definition, the local fractional derivative (LFD) is defined as ($0 < \alpha \leqslant 1$)

$$D^{\alpha} f(y) = \lim_{x \to y} \frac{\mathrm{d}^{\alpha}[f(x) - f(y)]}{[\mathrm{d}(x - y)]^{\alpha}} \tag{4}$$

This LFD was used to study local fractional differentiability properties of irregular functions. A geometrical interpretation, as the coefficient of a local power law, was also given owing to the fact that it appears naturally in local fractional Taylor series expansions. Later, simple local fractional differential equations (LFDE) involving LFDs were studied [15]. We remark that, because of the limit, the LFD is not an analytic continuation in order of the usual integer order derivatives. Perhaps this makes it suitable to study local scaling behavior.

Cornetti, in his PhD thesis [24], and later Carpinteri and Cornetti [14] used the simple LFDE and generalized the relation between the strain and the displacement in the case where the strain is localized on a fractal set. They proposed to use:

$$\epsilon^{*}(x) = D^{\alpha} u(x) \tag{5}$$

where ϵ^{*} is a renormalized strain and u is the displacement. Interestingly, this generalization of the classical relation between strain and displacement gives, in addition to a geometrical interpretation, also a physical interpretation to the LFD. In our opinion, this is the first important step towards possible generalization of classical continuum mechanics. It is important to note that the classical definition of fractional derivative in Eq. (2) can not be used to this purpose as the basic requirement of Galilean invariance would not be satisfied. In fact, if we add a constant to the displacement field, the stress field would change but this is not the case with the LFD. It should be pointed out that the relation (5) "detects" only the singular part of the displacement field even if there is overriding smooth displacement field. This follows from the fact that the LFD of smooth functions is zero. Therefore, this formalism automatically separates the singular part of the displacement field and "acts" on that.

The integral of the unit function over the interval $[0, 1]$ gives the length (the measure) of the interval $[0, 1]$. To extend this idea to the computation of the measure of fractal sets built on the real axis, it can be seen immediately that the fractional integral (1) does not work as it fails to be additive because of its non-trivial kernel. On the other hand, starting from Eq. (1), it was suggested in [16] that a definition of measure of a fractal set can be obtained through the *fractal integral* of order α of the function $f(x)$ over the interval $[a, b]$ defined as

$${}_{a}D_{b}^{-\alpha} f(x) = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(\tilde{x}_i) \frac{\mathrm{d}^{-\alpha} 1_{\mathrm{d}x_i}(x)}{[\mathrm{d}(x_{i+1} - x_i)]^{-\alpha}} \tag{6}$$

where $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$, $x_0 = a$ and $x_N = b$, provide a uniform partition of the interval $[a, b]$; $\tilde{x}_i$ is the point for which $f(x)$ attains the maximum value in the subinterval $[x_i, x_{i+1}]$, while $1_{\mathrm{d}x_i}(x)$ is the unit function defined on the same subinterval.

Interestingly, the fractal integral is infinite for a function which is constant throughout, whereas it can be finite and different from zero if $f(x)$ has a fractal support whose Hausdorff dimension d is equal to the fractional order of integration α. Denote with $f_A(x)$ the restriction of $f(x)$ to the fractal set A (i.e. $f_A(x) \equiv f(x)$ if $x \in A$, 0 otherwise) and consider, for instance, the triadic Cantor set C, built on the interval $[0, 1]$, whose dimension is $d = \ln 2/\ln 3$. $1_C(x)$ is the restriction of the unit function to this set. In order to compute its fractal integral of order α, we can apply Eq. (6) with $x_0 = 0$ and $x_N = 1$ and choose $\tilde{x}_i$ to be such that $1_C(\tilde{x}_i)$ is maximum in the interval $(x_i, x_{i+1}]$. This yields (see [17] for fractional integration of the unit function):

$${}_{0}D_{1}^{-\alpha} 1_C(x) = \lim_{N \to \infty} \sum_{i=0}^{N-1} F_C^i \frac{(x_{i+1} - x_i)^{\alpha}}{\Gamma(1 + \alpha)} \tag{7}$$

where F_C^i is a flag function that takes value 1 if the interval $(x_i, x_{i+1}]$ contains a point of the set C and 0 otherwise. It can be easily seen that ${}_{0}D_{1}^{-\alpha} 1_C(x)$ is infinite if $\alpha < d$, and 0 if $\alpha > d$. For $\alpha = d$, we find ${}_{0}D_{1}^{-\alpha} 1_C(x) = (1/\Gamma(1 + \alpha))$. Therefore ${}_{0}D_{1}^{-\alpha} 1_C(x)$ is a suitable (fractal) measure of the triadic Cantor set C. For this simple set this definition of measure yields the same value of the dimension as that of the Hausdorff one, the difference being only in the value of the normalization constant, $(1/\Gamma(1+d))$, instead of $(\Gamma^d(1/2)/\Gamma(1+d/2))$. In next sections we will always refer to the fractal measure defined by Eq. (7).

It can be seen that, if we considered the function ${}_0D_x^{-\alpha}1_C(x)$ in the case $\alpha = d$, it would be a non-decreasing Cantor staircase function taking a constant value in the intervals where there are no points of the Cantor set. It is bounded by two power laws: $bx^\alpha \leqslant {}_0D_x^{-\alpha}1_C(x) \leqslant ax^\alpha$.

A mathematically rigorous treatment of these integrals would involve considering different partitions and evaluating the function at arbitrary points inside the intervals. On the other hand, we decided to concentrate on applications and choose a natural partition given by the generation of the underlying fractal set and evaluate the function at the midpoint of the intervals of this partition.

3. Fractal integrals of basic functions on Cantor sets

In this section, we will compute fractal integrals of simple functions whose supports are represented by deterministic fractal sets. The order α of the fractal integrals will be always equal to the dimension ($\alpha \leqslant 1$) of the fractal support of the integrand function in order to obtain finite values—see Eq. (7). We are mainly interested in the results and their possible applications to solid mechanics (see Section 5).

Fractal integrals of basic functions will be performed over generalized Cantor sets. The triadic Cantor set built on the segment $[0,1]$ is obtained by removing the middle third at each step in the generation procedure [7]. For these reasons, hereafter it will be marked by $C_3^{[0,1]}$. On the other hand, one can decide to remove from the middle of the segment its fraction $1/p$ ($p \geqslant 1$). A fractal set is obtained with dimension α equal to:

$$\alpha = \frac{\ln 2}{\ln\left(\frac{2p}{p-1}\right)} \tag{8}$$

Respectively, for $p = 1, 3, \infty$, the fractal dimension is $\alpha = 0, \ln 2/\ln 3, 1$. In these cases we will indicate the fractal set by $C_p^{[0,1]}$ and by $C_p^{[a,b]}$ if it is built on the interval $[a,b]$. See Fig. 1 where we drew, for illustrative purposes, the triadic Cantor set $C_3^{[0,1]}$ and $C_2^{[0,1]}$.

For the sake of simplicity, we will mark the fractal integral (6) of order α over the Cantor set $C_p^{[a,b]}$ by the following notation:

$${}_aD_b^{-\alpha}f_{C_p^{[a,b]}}(x) \equiv \int_{C_p^{[a,b]}} f(x)\,\mathrm{d}^\alpha x \tag{9}$$

We indicate only with Γ the value $\Gamma(\alpha+1)$ of the Gamma function.

Now the goal is the computation of the fractal integrals of the following functions: $1, x, x^2$. In order to evaluate the integral at different refinement levels, we use the natural partitions given by the subsequent iterations generating the fractal set and we use the value of the function at the midpoint of the related intervals. Let us begin from the integral of the unit function over the triadic Cantor set by using the sequence of intervals used in the iterative procedure for building the Cantor set starting from the unit segment. As is well-known, at the nth step, the intervals Δx_n are 2^n ($i = 1,\ldots,2^n$) each one of length 3^{-n}. Since the integrand function is equal to one everywhere on the Cantor set, the application of the definition (6) to the partition just described yields the required result:

$$\int_{C_3^{[0,1]}} 1(x)\,\mathrm{d}^\alpha x = \frac{1}{\Gamma}\lim_{n\to\infty}\sum_{i=1}^{2^n}(\Delta x_n)^\alpha = \frac{1}{\Gamma} \tag{10}$$

which is the same as the one Eq. (7) provides. Note that the classical integral of the unit function over the triadic Cantor set would have been equal to zero since its length is null (like the length of every fractal set with dimension lower than one).

(a) (b)

Fig. 1. (a) Triadic Cantor set ($p = 3$, $\alpha = \ln 2/\ln 3$) and (b) generalized Cantor set ($p = 2$, $\alpha = 1/2$).

Next consider the fractal integrals of x and x^2: $\int_{C_3^{[0,1]}} x\,\mathrm{d}^\alpha x$ and $\int_{C_3^{[0,1]}} x^2\,\mathrm{d}^\alpha x$. Again we can refer to the iterative procedure. For the integration of x, the first two iterations ($n = 1, 2$) yield:

$$\begin{aligned} n = 1, \quad & \frac{1}{\Gamma}\left(\frac{1}{3}\right)^{\alpha}\left[\frac{1}{6}+\frac{5}{6}\right] = \frac{1}{2\Gamma} \\ n = 2, \quad & \frac{1}{\Gamma}\left(\frac{1}{3}\right)^{2\alpha}\left[\frac{1}{18}+\frac{5}{18}+\frac{13}{18}+\frac{17}{18}\right] = \frac{1}{2\Gamma} \end{aligned} \tag{11}$$

It can be seen that, due to the symmetry in the Cantor set we are considering, for every x_i in the square bracket above there exist $1 - x_i$. Therefore, the squared term is equal to 2^{n-1} and we get the same final result at any iteration. For the case of x^2 we need to use the symmetry and also the self-similarity of the set and then a straightforward calculation gives the result:

$$\begin{aligned} n = 1, \quad & \frac{1}{\Gamma}\left(\frac{1}{3}\right)^{\alpha}\left[\left(\frac{1}{6}\right)^2+\left(\frac{5}{6}\right)^2\right] = \frac{1}{\Gamma}\left(\frac{1}{4}+\frac{1}{3^2}\right) \\ n = 2, \quad & \frac{1}{\Gamma}\left(\frac{1}{3}\right)^{2\alpha}\left[\left(\frac{1}{18}\right)^2+\left(\frac{5}{18}\right)^2+\left(\frac{13}{18}\right)^2+\left(\frac{17}{18}\right)^2\right] = \frac{1}{\Gamma}\left(\frac{1}{4}+\frac{1}{3^2}+\frac{1}{3^4}\right) \end{aligned} \tag{12}$$

For n tending to infinity, we obtain the results we were looking for:

$$\int_{C_3^{[0,1]}} x\,\mathrm{d}^\alpha x = \frac{1}{2\Gamma} \tag{13}$$

$$\int_{C_3^{[0,1]}} x^2\,\mathrm{d}^\alpha x = \frac{1}{\Gamma}\left[\frac{1}{4}+\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^{2j}\right] = \frac{3}{8\Gamma} \tag{14}$$

As described above, as p goes from 1 to infinity, it is possible to build generalized Cantor sets $C_p^{[0,1]}$ showing a fractal dimension α varying continuously from 0 to 1 (Eq. (8)). Further computations are needed to extend the results (10), (13) and (14) to the integration upon generalized Cantor sets. Since the procedure is similar, we skip the details and state directly the results. Moreover, the integration performed on fractal sets $C_p^{[0,b]}$ built on intervals other than $[0, 1]$ can be easily obtained by applying the scaling property or by explicit calculations.

$$\int_{C_p^{[0,b]}} 1\,\mathrm{d}^\alpha x = \frac{b^\alpha}{\Gamma} \tag{15}$$

$$\int_{C_p^{[0,b]}} x\,\mathrm{d}^\alpha x = \frac{b^{1+\alpha}}{2\Gamma} \tag{16}$$

$$\int_{C_p^{[0,b]}} x^2\,\mathrm{d}^\alpha x = \frac{b^{2+\alpha}}{\Gamma}\left\{\frac{1}{4}+\left[\frac{p+1}{2(p-1)}\right]^2\sum_{j=1}^{\infty}\left(\frac{p-1}{2p}\right)^{2j}\right\} = \frac{pb^{2+\alpha}}{(3p-1)\Gamma} = \frac{b^{2+\alpha}}{2(1+2^{-1/\alpha})\Gamma} \tag{17}$$

Note that, for $\alpha = 1$, the classical results $b, b^2/2, b^3/3$ are easily recovered respectively for Eq. (15)–(17). Note that Eq. (15) provides the (fractal) measure $b^* = b^\alpha/\Gamma$ of $C_p^{[0,b]}$.

There are two types of scaling possible in the above integrals. In one case, the underlying fractal set remains the same but only the integration limit b is changed. In the other case, all the points of the fractal set are scaled along with the limit of integration. In the first case we get a staircase like singularly continuous function, whereas in the second case it would be a smooth power law which is nothing but an envelop of the staircase functions of the first case. Both kinds of situations can arise in solid mechanics. The former one could describe the displacement field inside a stretched body, whereas the latter quantity should be needed to find the total displacement of the body under stretching.

The above results have a clear physical meaning in the framework of statics. They allow the first principle calculation of the resultant and the resultant moment of a system of forces linearly distributed on a fractal set. It is similar to the usual computation in classical solid mechanics, although the dimensions of the involved physical and geometrical quantities are anomalous. In order to generalize to fractal sets the concept of stress, we define a fractal stress $\mathbf{t}^*$ given by the limit

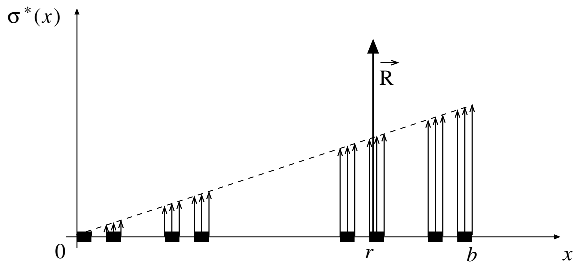

Fig. 2. Linear fractal stress distribution upon a Cantor set: the resultant $\vec{R}$ and its arm r.

$$\mathbf{t}^* = \lim_{\Delta A^* \to 0} \frac{\Delta \mathbf{F}}{\Delta A^*} \tag{18}$$

where $\Delta\mathbf{F}$ is the force acting upon the fractal subset of fractal measure ΔA^*. See [4,13,25] for further details. For the sake of simplicity, here we will consider only fractal sets lying on the x-axis and the component $\sigma^*(x)$ of the fractal stress $\mathbf{t}^*$ perpendicular to it (Fig. 2). Clearly the need to define such a quantity stems from the fact that we have taken ideal fractal sets without any lower cutoff. Its modification to incorporate such a cutoff is easily possible but that is not our aim here. Therefore, a load distributed on a fractal set gives rise to a finite fractal stress, while the length of the fractal set and the stress upon it are respectively zero and infinite, and therefore not of any use [4]. Consider now, for instance, a linearly increasing fractal stress acting upon a generalized Cantor set $C_p^{[0,b]}$ (henceforth we assume unit thickness) as represented in Fig. 2 (where we chose $p = 3$ and, of course, we drew a prefractal stress distribution). If $\bar{\sigma}^*$ is the maximum value, its value at the origin being zero, the fractal stress distribution is described by $\sigma^*(x) = (\bar{\sigma}^*/b)x$. Fractal integrals permit the computation of the resultant R and of the resultant moment M around the point O:

$$R = \int_{C_p^{[0,b]}} \sigma^*(x)\,\mathrm{d}^\alpha x = \int_{C_p^{[0,b]}} \frac{\bar{\sigma}^*}{b} x\,\mathrm{d}^\alpha x = \frac{\bar{\sigma}^* b^\alpha}{2\Gamma} \tag{19}$$

$$M = \int_{C_p^{[0,b]}} x\sigma^*(x)\,\mathrm{d}^\alpha x = \int_{C_p^{[0,b]}} x^2 \frac{\bar{\sigma}^*}{b}\,\mathrm{d}^\alpha x = \frac{\bar{\sigma}^* b^{1+\alpha}}{2(1+2^{-1/\alpha})\Gamma} \tag{20}$$

In order to reduce the system of distributed fractal stresses to a unique force, its modulus is R and its arm r from the origin is given by

$$r = \frac{M}{R} = \frac{1}{1+2^{-1/\alpha}} b \tag{21}$$

For α equal to 1, the classical value $r = 2/3b$ is recovered, while, in the case of a fractal stress distribution over the triadic Cantor set such as the one drawn in Fig. 2, we find $r = 3/4b$.

It is clear that the prefactors in Eqs. (19) and (20) would also depend on the lacunarity of the fractal set chosen and they are specific to the example chosen here. On the other hand, we hope that the prefactor in Eq. (21) could have more ‘‘universal’’ character, since it comes from the ratio of two quantities which involve the same fractal set. This remark would apply to some further calculations too.

4. An example of material with fractal microstructure: concrete

Before applying the results of the previous section to the computation of the strength of structures made by a material with a fractal microstructure, we intend to show an example of such a material: concrete. Fractal patterns in the failure process of concrete structures have been experimentally detected in several experiments (see, for instance, [26–28]). Here we want to give a theoretical explanation of these fractal patterns based on the aggregate grading inside concrete.

Concrete is composed by aggregates, i.e. stones of different dimensions (sand, gravel), which take place in a cement matrix. The volume percentage of the aggregates is about 75% of the total volume. For what concerns the tensile behavior, the weakest link in normal strength concrete is represented by the interface between cement matrix and aggregates [29]. Hence, for loads close to the peak, when damage concentrates, microcracks develop all around the aggregates. The final macrocrack leading to failure, in normal strength concrete, is usually intergranular (i.e. no grain

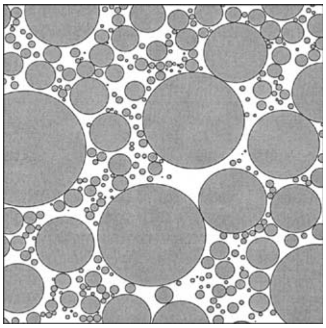

Fig. 3. Scheme of the cross section of a concrete specimen characterized by a Füller mix (from [30]): the circles are the interceptions of the grains with the section, while the remaining part is occupied by the cement matrix.

breaks). In other words, considering the cross section where the main tensile crack will develop at the end of the failure process, the stress is mainly carried by the cement matrix (Fig. 3). It is therefore reasonable to look for a link between the aggregate grading and the features of the set where stress concentrates.

Thanks to its good packing property, the most common aggregate size distribution used to prepare concrete is the so-called Füller mix. The Füller mix [30] can be expressed in terms of the grain size distribution function as follows:

$$f(d) = 2.5\frac{d_{\min}^{2.5}}{d^{3.5}}, \quad d_{\min} \leqslant d \leqslant d_{\max} \tag{22}$$

where we assumed the aggregates to be spherical with diameter d; $d_{\max}$ and $d_{\min}$ are respectively the maximum and minimum diameter, usually equal to 30 mm (but it can increase up to 120 mm in dams) and 0.2 mm. $f(d)$ is a probability density function, i.e. $f(d)\,\mathrm{d}d$ is the fraction of grains with diameter belonging to the interval $[d, d+\mathrm{d}d]$ and $\int_{d_{\min}}^{d_{\max}} f(d)\,\mathrm{d}d = 1$. Eq. (22) shows clearly that the number of the small particles is higher than that of the large particles, since the former ones must fill the gaps between the latter ones.

Based on the experimental evidence that the matrix–aggregate interface is the weakest link inside concrete and therefore negligible, we want now to analyse the fractal features of the stress carrying ligament (thick line) of the concrete specimen 2D scheme of Fig. 4a. Let us mark with l the generic intersection of the particle of diameter d with the ligament $\overline{BB'}$. Let the probability density function of the interceptions l be $s(l)$. A stereological analysis allows us to obtain $s(l)$ from Eq. (22):

$$s(l) = \begin{cases} \frac{2l}{\overline{d_V^2}}\left[\left(\frac{d_{\min}}{l}\right)^{2.5} - \left(\frac{d_{\min}}{d_{\max}}\right)^{2.5}\right], & d_{\min} \leqslant l \leqslant d_{\max} \\ \frac{2l}{\overline{d_V^2}}, & 0 < l < d_{\min} \end{cases} \tag{23}$$

where $\overline{d_V^2}$ is the second moment of the distribution (22). Therefore, in a wide range of scales, $s(l)$ can be approximated by a power law with negative exponent 1.5: $s(l) \sim l^{-1.5}$. This means that the resistant ligament can be modeled by a

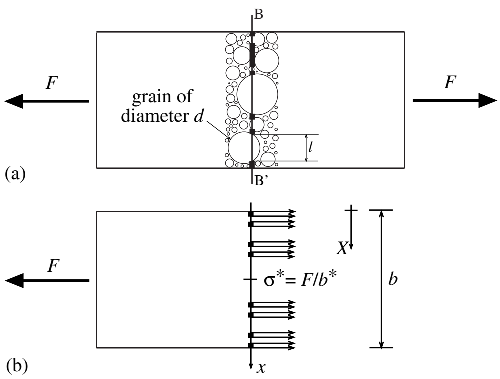

Fig. 4. Concrete specimen under a tensile load F (a); uniform fractal stress distribution upon the resistant ligament (b).

lacunar fractal set (see Fig. 4b) of dimension $\alpha = 0.5$. Of course this would be true if the aggregates filled the whole concrete bulk. Nevertheless, the high value of the aggregate volume fraction used in concrete casting suggest us to approximate the ligament as a fractal set of dimension α when dealing with a tensile failure.

5. Tensile and flexural strength of disordered material specimens

Let us consider a specimen composed by a concrete-like or disordered material. We wish to compute its tensile and flexural strength from first principles. The resistant ligament is modeled by a lacunar fractal set of dimension α. In order to use the result of Section 3, we will assume this fractal set to be a generalized Cantor set (Fig. 1).

Let us start with the case of the specimen subjected to a tensile load F (Fig. 4b). Making the assumption of constant fractal stress all over the ligament, the equilibrium along the specimen axis is satisfied if:

$$F = \int_{C_p^{[-b/2,+b/2]}} \sigma^* \, \mathrm{d}^\alpha x = \sigma^* \int_{C_p^{[0,b]}} \mathrm{d}^\alpha X = \frac{\sigma^* b^\alpha}{\Gamma} \Rightarrow \sigma^* = \frac{F}{b^*} \tag{24}$$

where we performed the change of variable $X = x + b/2$ in order to use the result of Eq. (15); $b^* = b^\alpha/\Gamma$ is the fractal measure of the ligament as defined in Section 3. According to Eq. (24), the fractal stress value increases linearly with the load. Assuming a (fractal) stress resistance criterion, rupture takes place when σ^* reaches the critical value σ_u^*, which is therefore called *fractal strength*. σ_u^* is exclusively a property of the material, i.e. it is independent of the specimen size and of the test geometry. The corresponding ultimate load F_u is therefore equal to:

$$F_u = \sigma_u^* b^* = \frac{\sigma_u^*}{\Gamma} b^\alpha \tag{25}$$

Eq. (25) represents the scaling law of the critical load, that is, its dependence on the size b of the specimen. If we consider the *nominal* strength $\sigma_u = F_u/b$ as responsible for the rupture, we will find that σ_u is no longer an exclusive property of the material but it is size-dependent. In fact, from Eq. (25), we find:

$$(\sigma_u)_{\text{tensile}} = \frac{\sigma_u^*}{\Gamma} b^{-(1-\alpha)} \tag{26}$$

The subscript tensile reminds that it is the nominal strength calculated in direct tension tests. Eq. (26) is consistent with the well-known decrease of strength as the structural size increases and was firstly derived by Carpinteri [4] in a slightly different form. Several authors experimentally revealed this peculiar behavior, which is one of the most important size effects in material science. For what concerns the absolute value of the power law exponent in Eq. (26), Researchers always found values comprised in the interval $[0, 1/2]$. This range agrees with the value $\alpha = 0.5$ computed in Section 4, assuming that all the matrix–aggregate interfaces fail before the peak load is reached.

Eventually observe that, if the dimension α of the ligament is known, we can calculate the fractal strength directly from Eq. (25); otherwise we can compute the nominal strength for specimens of different size and obtain the fractal strength (and the fractal dimension) interpolating the data according to Eq. (26).

Consider then the same specimen as before, now subjected to a bending moment M (Fig. 5a). Its fractal strength σ_u^* is known, for instance, from direct tension tests as described above. The aim is to compute the ultimate moment M_u in the hypothesis that the beam will break when the fractal stress reaches somewhere its critical value σ_u^*. For the sake of simplicity, we assume the stress distribution to be linear and fractal both in tension and compression, as shown in Fig. 5b. The fractal stress is therefore zero on the specimen axis. At rupture, it can be analytically expressed by

$$\sigma^*(x) = \frac{\sigma_u^*}{b/2} x \tag{27}$$

The resultant is null, while the resultant moment is

$$M_u = \int_{C_p^{[-b/2,+b/2]}} x \sigma^* \, \mathrm{d}^\alpha x = \frac{2\sigma_u^*}{b} \int_{C_p^{[-b/2,+b/2]}} x^2 \, \mathrm{d}^\alpha x \tag{28}$$

Upon substitution of x with $X = x + b/2$, we can apply Eqs. (15)–(17) to get

$$M_u = \frac{2\sigma_u^*}{b} \int_{C_p^{[0,b]}} \left(X - \frac{b}{2} \right)^2 \mathrm{d}^\alpha X = \frac{\sigma_u^* b^{1+\alpha}}{2\Gamma} \left(\frac{2^{1/\alpha} - 1}{2^{1/\alpha} + 1} \right) \tag{29}$$

Since the exponent α is lower than 1, the ultimate moment increases with the size more slowly than the classical beam theory predicts (b^2). As in the tensile case, the classical definition of stress is meaningless in the case of bodies with

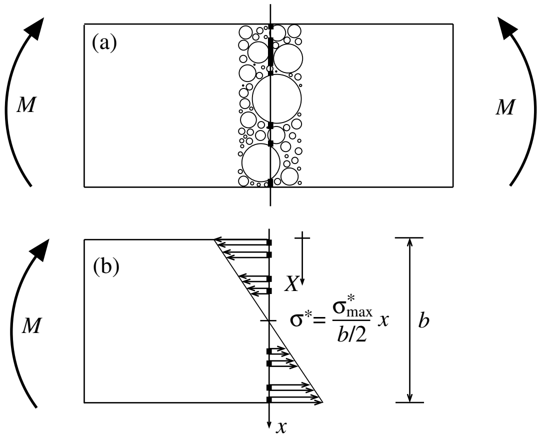

Fig. 5. Concrete specimen subjected to a bending moment M (a); linear fractal stress distribution upon the resistant ligament (b).

fractal microstructure; if one considers the nominal strength $\sigma_u = 6M_u/b^2$ as responsible for rupture, he will find that σ_u is no longer a material property but it is a function of the size

$$(\sigma_u)_{\text{flexural}} = 3\left(\frac{2^{1/\alpha} - 1}{2^{1/\alpha} + 1}\right)\frac{\sigma_u^*}{\Gamma} b^{-(1-\alpha)} \tag{30}$$

The subscript flexural reminds that it is the nominal strength calculated in bending tests. Eq. (30) provides the size effect affecting the flexural strength. As in tensile tests, the nominal strength decreases as the size increases. We notice that the power law exponent is the same as in Eq. (26); nevertheless Eq. (30) differs from the previous one because of the presence of a numerical coefficient whose value varies from 1 to 3 as α varies from 1 to 0. While the scaling exponent is a function only of the fractal dimension of the ligament, the numerical coefficient is affected also by the fractal set describing the ligament itself. Comparing Eqs. (26) and (30), we find that, in the particular case of a Cantor set-like ligament, the nominal flexural and tensile strengths are linked through the relationship:

$$(\sigma_u)_{\text{flexural}} = 3\left(\frac{2^{1/\alpha} - 1}{2^{1/\alpha} + 1}\right)(\sigma_u)_{\text{tensile}} \tag{31}$$

from which it appears that the nominal flexural strength is always higher than the tensile one (as shown by experiments), except in the Euclidean case $\alpha = 1$, when they are equal. The result (31) is a consequence of the kind of fractal chosen to describe the cross section. However, what is important is that, while the fractal strength σ_u^* is an exclusive property of the material, the nominal strength depends not only on the material but also on the size, as remarked by Carpinteri et al. [31], and on the test geometry, as Eq. (31) clearly shows.

6. Conclusions

We considered the problem of generalizing classical continuum mechanics to materials with fractal microstructure. We used recently introduced concepts of local fractional derivative and fractal integrals. It was already shown that a simple equation involving LFD can describe the relation between strain and displacement in the cases where the stress is concentrated on a fractal set. Here, on one hand, we demonstrate one origin of such a stress concentration on fractal sets, viz, the heterogeneity of the aggregates in the concrete. On the other hand, given the existence of the concentration of the stress on a fractal set, we develop a way to make first principle calculations of various strengths. For this purpose we use the concept of fractal integrals. We hope that these calculations will pave the way for more general treatment of these questions. Future work can proceed in various directions such as making these concepts mathematically rigorous and generalizing them to dimensions greater than one.

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