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Buckling instability of Vlasov thin-walled open-section beams: The Euler–Prandtl coupled problem

Alberto Carpinteri [a,b], Giuseppe Nitti [a,*]

[a] Politecnico di Torino, Department of Structural, Geotechnical and Building Engineering, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
[b] Shantou University, Department of Civil and Environmental Engineering, Zhujiang (Pearl River), Professor of Guangdong Province, Shantou, PR China

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ABSTRACT

In the field of structural mechanics, the study of elastic stability is a very important stage in the design of slender, thin, and/or shallow elements. The loss of stability can lead to the global collapse of the structural element. This is achieved when the load applied to the structure reaches the critical value.

There are two fundamental instabilities in rectilinear beams: the axial buckling, due to the axial force, and the lateral–torsional buckling, due to the bending moment. In general, these two instabilities may interact, when both axial force and bending moments are applied at the same time.

The present paper will illustrate a three-dimensional coupled formulation, including Vlasov's Theory for thin-walled open-section beams. The compactness of the final expressions, made even more evident by the matrix formulation, allows us for a very convenient use in the case of automatic computations.

1. Introduction

The phenomenon of instability is a topic that has received a great attention from the scientific community. The first contributions date back to the 18th and 19th Centuries. Even today we often make reference to problems introduced by Euler [1] (axial buckling) and Prandtl [2] (lateral–torsional buckling). On the other hand, the attempts to couple the two problems are relatively few and incomplete.

The first study on elastic instability is reported in Euler's Treatise of 1759 [1], in which axial buckling and critical axial load, leading to the loss of stability of a rectilinear beam, are defined. In 1899 Michell [3] deduced that the loss of stability of a beam can also be due to its lack of flexural and torsional rigidity. Around the same time, Prandtl [2] proposed a theory that describes the loss of stability by flexion–torsion perturbation of a slender and/or deep beam subject to an uniform bending moment. In the mid-30s of last century, the German engineer Wagner published a work (first in German [4], and later in English [5]), in which he provided the equations for the determination of the critical forces of torsional instability of thin-walled open-section beams present in aircraft structures. In his formulation, in order to determine the axial stresses due to twisting, Wagner used a relationship similar to that of sectorial areas introduced by Vlasov in 1936 [6]. In the same years, Znamenskii published an article [7] in which he used the Ritz–Timoshenko method to obtain approximate expressions of the critical torsional force. It should also be noted that, examining torsional deformations, both Wagner and Znamenskii assumed the torsional center coinciding with the shear center. In fact, it was demonstrated a few years later by Vlasov himself [8]. The experiments conducted by Boloban in 1936 [9] on aircraft spars showed that, in beams subject to torsion, axial instability occurs for critical forces considerably lower than the theoretical values obtained from Euler's formulation. In 1936 F. and H. Bleich published a paper [10] devoted to the problem of torsion and stability of thin-walled open-section beams. Using an energy method to describe the problem, the Authors obtained a system of three differential equations. On the other hand, they assumed that the sections remain plane after deformation. Always in those years, the scientists who contributed mostly to the general instability theory of thin-walled open-section beams were Bleich [11], Timoshenko [12,13], and Vlasov [8,14]. In particular, Bleich used a procedure describing the total potential energy of the beam as the difference between its deformation energy and the work by external loads. Timoshenko instead used the static method, that is, he wrote the equilibrium equations of the forces in the deformed shape configuration, whereas Vlasov transformed the stresses generated in a beam into fictitious external loads. Many years later, Anderson [15] and Attard [16] carried out experimental studies on mono-symmetric thin-walled open-section cantilever beams that confirmed the results obtained using the analytical formulations proposed by Timoshenko and Vlasov. Over the years, several efforts have been spent to rewrite the stability theory of thin-walled open-section beams. To describe the problem and on the basis of the theories by Bleich and Vlasov, different researchers used an energy approach. This method is the best to analyze the local stability of a structural element, but it involves excessive

* Corresponding author.
E-mail address: giuseppe.nitti@polito.it (G. Nitti).

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calculations when applied to more complex structures. Among others, we can mention the works by Ghobarah [17] and Roberts [18]: they took up Vlasov's Theory by removing the hypothesis of annihilation of shearing stresses on the mid-line of the section. Yang and McGuire [19], and Kitipornchai and Chan [20] improved the deformation energy considering all linear and nonlinear terms. In the first of these studies, the beams with doubly symmetrical section are analyzed, whereas in the second, L- and T-section beams (cross-section without warping) are analyzed. Although greatly complicating the calculations, their results are confirmed as special cases of the classical Euler's theory. In 1985, on the basis of the general theory of elastic stability due to Koiter [21–23], Pignataro et al. [24] performed a post-buckling analysis of simply supported channel beams. This technique was subsequently improved and the initial imperfection effects were taken into account [25]. In the 1990s, Pi [26] wrote a new formulation considering the rotation components of the second order, whereas Trahair [27] rewrote the deformation energy in a simplified form considering only the linear terms. Ronagh [28,29], on the other hand, studied the instability of the open-section beams with variable section, whereas Mohri [30] studied the post-buckling behavior of the same elements. In 2016 Taig et al. [31] presented an analytical approach for stability analysis of thin-walled beams implemented within the framework of the Generalized Beam Theory [32,33], a one-dimensional theory used in structural mechanics. In this manner, it was possible to account for the deformability of the cross-section in both pre-buckling and buckling analysis. More recently, Tong and Zhang [34,35] performed comparative simulations of beams under different load conditions by using Finite Element models (FEM).

In the present paper, an analytical formulation, based on the lecture 12 of the course "Static and Dynamic Instability of Structures" held by Carpinteri (private communication [36]) and the PhD thesis by Nitti [37], is presented, which allows the stability analysis of a thin-walled open-section beam. A coupling of Euler's Theory (axial buckling) and of Prandtl's Theory (lateral–torsional buckling) is introduced. Furthermore, an energy-based method is defined that allows determining the equation of non-uniform torsion by Vlasov [38]. Finally, the equation of non-uniform torsion with lateral–torsional buckling effects is also analytically defined, that provides the value of the critical bending moment for thin-walled open-section beams [39]. The analytical formulation that will be presented in this paper differs from similar formulations obtained in the previous years by other Authors. In this regard, both Vlasov [14] and Timoshenko [13] do not consider the work of deformation given by the bimoment. On the other hand, the latter has been considered by Pi [26] and Zhang [35], only in the case of I-beam sections with double symmetry. However, differently from the last two papers, where only the flexural stress is considered, the effect of stress due to axial force is also considered herein. In the papers by Pignataro et al. [24] and Piccardo [32], a rather complex formulation is presented, in which the assumption of non-deformability of sections is removed, with a focus onto post-buckling analysis of simply supported channel beams.

2. A brief outline of Vlasov's theory

In the second half of the 1930s, Vlasov published some papers [6,40, 41], in which a new method to determine strains and stresses in thin-walled open-section beams is illustrated. This approach, called *Sectorial Areas' Theory*, is an extension of Saint Venant's Theory. A few years later, the same Author extended further his theory [8] introducing a new characteristic of the internal loading: the *bimoment*. In 1945, Timoshenko published an extensive paper [42] in which he analyzes the open-section thin beam subject to torsional moment. The analytical formulation, although presenting a different notation, has extensive references to Vlasov's Theory [8]. This work had a remarkable international success, since it was published in English (unlike the papers by Vlasov, which were published in Russian) and thus accessible to more researchers. From this moment on, the interest in the topic

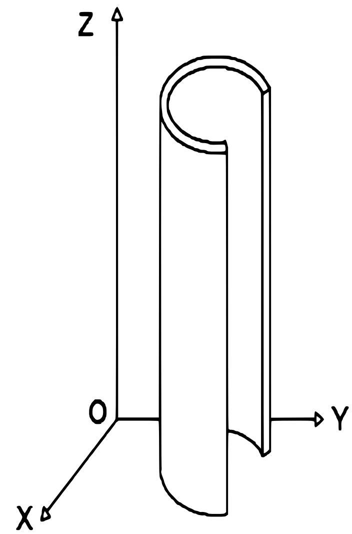

Fig. 1. Thin-walled open-section beam.

increased significantly, to such an extent that The Israeli Program for Scientific Translations, a government company focused onto translation and publication of scientific and technical manuscripts from Russian to English, recognized the importance of the work by Vlasov (who died on August 7, 1958), publishing in 1961 the English translation of his 1940 volume. This book, Thin-walled Elastic Beams [14], remains a milestone in the scientific literature. The formulation of Vlasov's Theory has already been widely reported in various papers [38,43] and in the book [44], and it is only summarized in the following.

Let us consider a thin-walled open-section beam, without symmetry axes, located in a coordinate system with origin in the shear center of the section ($I_{x\omega} = I_{\omega x} = I_{y\omega} = I_{\omega y} = 0$) with axes X and Y oriented according to the principal directions ($I_{xy} = I_{yx} = 0$), whereas the Z axis is parallel to the longitudinal centroidal axis of the beam (Fig. 1). In addition, considering that the sectorial area is evaluated with respect to the sectorial centroid, the sectorial static moment S_ω is zero by definition [14,45]. The deformed shape can be defined by means of three independent variables corresponding to the three generalized displacements of the cross-section: two translations ξ and η in the X and Y directions, respectively, and the rotation angle ϑ around the Z axis. Based on the Superposition Principle, the total axial deformation can be reduced to the sum of four different contributions: one is purely axial, two are purely flexural, whilst the last is provided by the bimoment [46]. In the last case, the section does not remain plane (warping effect) and produces an additional axial stress with respect to those due to axial force and flexural moments [47,48]. This is related to the warping of the section [49]. The intensity of this stress state cannot be neglected for thin-walled open-section beams and the simple application of Saint Venant's Theory could lead to gross errors [50]. In analytical terms, the bending moments, M_x and M_y, and the bimoment B are written as:

$$M_x = N y_G - E I_{xx} \eta'' \tag{1a}$$

$$M_y = N x_G - E I_{yy} \xi'' \tag{1b}$$

$$B = -E I_{\omega\omega} \vartheta'' \tag{1c}$$

where E is the elastic modulus, I_{xx} and I_{yy} are the moments of inertia, $I_{\omega\omega}$ is the sectorial moment of inertia, while the terms x_G and y_G indicate the coordinates of the geometric center of gravity of the section evaluated with respect to its shear center. As can be observed, the presence of the axial force N does not allow a perfect diagonalization of

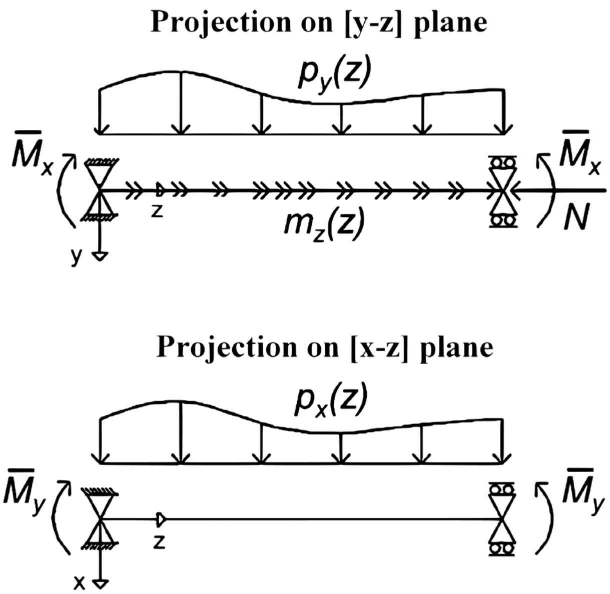

Fig. 2. Loading system applied to the beam.

Vlasov's Equations [38]. Finally, consider that Eq. (1b) would present opposite algebraic sign on the right-hand side due to the assumed positive directions of the axes and to the positive anti-clockwise moments. Nevertheless, we will adopt the conventions by Vlasov, which consider positive bending moments when stretching the intrados.

3. Analytical formulation for stability analysis

Consider a slender thin-walled open-section beam, without symmetry axes, constrained at both ends by cylindrical hinges. The right-handed reference system is used, with origin in the shear center of the section, and X, Y axes oriented according to the principal directions (i.e. the tensor of the moments of inertia, evaluated with respect to the reference system with origin in the shear center, is a diagonal matrix) and with the Z axis parallel to the longitudinal centroidal axis of the beam. It is pointed out that all parameters (static moments, moments of inertia, etc.), as well as the loading directions and stresses mentioned in the following equations, are evaluated with respect to this reference system. The loading system applied to the beam consists of two distributed orthogonal forces $p_x(z)$ and $p_y(z)$, and by one distributed torsional moment $m_z(z)$ along the axis of the beam. Furthermore, one axial force N and two bending moments $\bar{M}_x$ and $\bar{M}_y$ are applied at the ends of the beam, as shown in Fig. 2. The assumptions behind the formulation are the following:

- the shape of the cross-section remains unchanged after deformation;
- the element is deformable only by bending and twisting;
- Vlasov's Theory is valid.

In the present formulation, the variational energy method is applied, by which it is possible to determine the critical (metastable) equilibrium conditions of the system by the annihilation of the variation of the total potential energy of the beam (difference between the elastic deformation energy and the work by external loads). By applying Clapeyron's Theorem and considering that the bending moments and the bimoment are energetically orthogonal to each other, the elastic deformation energy Φ can be written as:

$$\Phi = \frac{1}{2}\int_0^L \left[M_y\xi'' + M_x\eta'' + B\vartheta'' + M_z\vartheta'\right]\mathrm{d}l \tag{2}$$

As reported previously, the work of deformation given by bimoment and torsional moment has also been considered in this equation, something that other Authors have neglected [13,14]. The primary torsional moment M_z, according to Saint Venant's Theory, can be written as:

$$M_z = \int_A \{r\}\wedge\{\tau_z\}\mathrm{d}A = \int_A \left(x\tau_{zy} - y\tau_{zx}\right)\mathrm{d}A \tag{3}$$

where τ_{zy} and τ_{zx} are the shearing stress components. Applying the elastic constitutive equations [51], it is possible to obtain:

$$M_z = GI_t\vartheta' \tag{4}$$

where G is the shear elastic modulus, and I_t is the torsional stiffness factor. Inserting Eqs. (1) and (4) into Eq. (2), the elastic deformation energy Φ takes the following form:

$$\begin{aligned}\Phi = \frac{1}{2}\int_0^L \Bigg[& Nx_G\xi'' - EI_{yy}\xi''^2 + Ny_G\eta'' - EI_{xx}\eta''^2 - EI_{\omega\omega}\vartheta''^2 + \\ & + GI_t\vartheta'^2\Bigg]\mathrm{d}l\end{aligned} \tag{5}$$

Its variation is equal to:

$$\begin{aligned}\delta\Phi = \int_0^L \Bigg[& \frac{1}{2}Nx_G\delta\xi'' - EI_{yy}\xi''\delta\xi'' + \frac{1}{2}Ny_G\delta\eta'' - EI_{xx}\eta''\delta\eta'' + \\ & - EI_{\omega\omega}\vartheta''\delta\vartheta'' + GI_t\vartheta'\delta\vartheta'\Bigg]\mathrm{d}l\end{aligned} \tag{6}$$

which integrated by parts provides:

$$\begin{aligned}\delta\Phi = & \Bigg[\frac{1}{2}N\left(x_G\delta\xi' + y_G\delta\eta'\right) - EI_{yy}\left(\xi''\delta\xi' - \xi'''\delta\xi\right) + \\ & - EI_{xx}\left(\eta''\delta\eta' - \eta'''\delta\eta\right) - EI_{\omega\omega}\left(\vartheta''\delta\vartheta' - \vartheta'''\delta\vartheta\right) + GI_t\vartheta'\delta\vartheta\Bigg]_0^L + \\ & + \int_0^L \left(EI_{yy}\xi^{IV}\delta\xi + EI_{xx}\eta^{IV}\delta\eta + EI_{\omega\omega}\vartheta^{IV}\delta\vartheta - GI_t\vartheta''\delta\vartheta\right)\mathrm{d}l\end{aligned} \tag{7}$$

To evaluate the work by concentrated loads N, $\bar{M}_x$, and $\bar{M}_y$, an axial strip of the beam with an infinitesimal area $\mathrm{d}A$ is considered, as shown in Fig. 3. By replacing the loads with a statically equivalent stress field, evaluated by Navier's formula, it is possible to write the equation of the work done by external forces in relation to an infinitesimal strip of beam. Integrating on the entire area, the total work by concentrated loads is obtained. With reference to the generic beam strip, whose position is identified by the coordinates (x, y), the axial stress σ_z, equivalent to a concentrated external load applied at the end of the beam, is equal to:

$$\sigma_z = \frac{N}{A} + \frac{\bar{M}_x}{I_{xx}}\left(y - y_G\right) + \frac{\bar{M}_y}{I_{yy}}\left(x - x_G\right) \tag{8}$$

The total work by the concentrated external loads can be written as:

$$L_e = \int_A \left(\sigma_z \Delta L\right)\mathrm{d}A = \int_A \sigma_z\left[\int_0^L \varepsilon_{eq}\mathrm{d}l\right]\mathrm{d}A \tag{9}$$

where the term

$$\varepsilon_{eq}\mathrm{d}l = \mathrm{d}l - \mathrm{d}z \tag{10}$$

expresses the axial displacement of the point of application of the force N due to the beam inflection with respect to the x and y axes, as shown in Fig. 4. The displacement

$$\Delta L = \int_0^L \varepsilon_{eq}\mathrm{d}l \tag{11}$$

can be evaluated as a function of the first derivatives of transverse displacements ξ and η, and must be calculated as the vectorial sum of the two displacement components ΔL_{yz} and ΔL_{xz} on the $[yz]$ and $[xz]$ planes:

$$\Delta L = \Delta L_{yz} + \Delta L_{xz} \tag{12}$$

The displacement component ΔL_{yz} can be written as:

$$\Delta L_{yz} = \mathrm{d}l - \mathrm{d}z_y = \mathrm{d}l - \mathrm{d}l\cos\varphi_x = \left(1 - \cos\varphi_x\right)\mathrm{d}l \tag{13}$$

Expanding the $\cos\varphi_x$ function in the Taylor series, we get:

$$\cos\varphi_x \cong 1 - \frac{\varphi_x^2}{2} \tag{14}$$

Carrying out similar considerations also for the displacement component ΔL_{xz} and performing the appropriate substitutions, the displacement components can be rewritten in the following form:

$$\Delta L_{yz} \cong \left(1 - 1 + \frac{\varphi_x^2}{2}\right)\mathrm{d}l = \frac{1}{2}\varphi_x^2\mathrm{d}l = \frac{1}{2}\xi'^2\mathrm{d}l \tag{15a}$$

$$\Delta L_{xz} \cong \left(1 - 1 + \frac{\varphi_y^2}{2}\right)\mathrm{d}l = \frac{1}{2}\varphi_y^2\mathrm{d}l = \frac{1}{2}\eta'^2\mathrm{d}l \tag{15b}$$

Finally, by replacing Eqs. (15) into Eq. (12), the total displacement can be written as:

$$\Delta L = \frac{1}{2}\left[\xi'^2 + \eta'^2\right]\mathrm{d}l \tag{16}$$

Since we are considering a generic beam strip, which does not commonly correspond to the shear center of the section, Eq. (16) then generalizes as:

$$\Delta L = \frac{1}{2}\left[\left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2\right]\mathrm{d}l \tag{17}$$

in which the displacements u and v are respectively:

$$u = \xi - \vartheta y \tag{18a}$$

$$v = \eta + \vartheta x \tag{18b}$$

The axial displacement of the point of application of the axial force N can be written as:

$$\begin{aligned}\Delta L &= \frac{1}{2}\left[\left(\xi' - \vartheta' y\right)^2 + \left(\eta' + \vartheta' x\right)^2\right]\mathrm{d}l = \\ &= \left[\frac{1}{2}\left(\xi'^2 + \eta'^2\right) + \frac{1}{2}\left(y^2 + x^2\right)\vartheta'^2 - \xi'\vartheta' y + \eta'\vartheta' x\right]\mathrm{d}l\end{aligned} \tag{19}$$

By inserting Eqs. (19) and (8) into Eq. (9), the deformation work due to concentrated external loads N, $\bar{M}_x$, and $\bar{M}_y$, can be written as:

$$\begin{aligned}L_e = \int_0^L \int_A \Bigg[&\left(\frac{N}{A} + \frac{\bar{M}_x}{I_{xx}}\left(y - y_G\right) + \frac{\bar{M}_y}{I_{yy}}\left(x - x_G\right)\right)\times \\ &\times\left(\frac{1}{2}\left(\xi'^2 + \eta'^2\right) + \frac{1}{2}\left(y^2 + x^2\right)\vartheta'^2 - \xi'\vartheta' y + \eta'\vartheta' x\right)\Bigg]\mathrm{d}A\mathrm{d}l\end{aligned} \tag{20}$$

For simplicity of exposure, the contributions of N, $\bar{M}_x$, and $\bar{M}_y$, are calculated separately, obtaining the following results.

The work by the axial force N is:

$$\begin{aligned}L_N &= \int_0^L \Bigg[\int_A \frac{1}{2}\frac{N}{A}\left(\xi'^2 + \eta'^2\right)\mathrm{d}A + \int_A \frac{1}{2}\frac{N}{A}\vartheta'^2\left(y^2 + x^2\right)\mathrm{d}A + \\ &\quad - \int_A \frac{N}{A}\xi'\vartheta' y\mathrm{d}A + \int_A \frac{N}{A}\eta'\vartheta' x\mathrm{d}A\Bigg]\mathrm{d}l = \\ &= \int_0^L \left[\frac{N}{2}\left(\xi'^2 + \eta'^2\right) + \frac{N}{2A}I_P\vartheta'^2 - \frac{N}{A}S_x\xi'\vartheta' + \frac{N}{A}S_y\eta'\vartheta'\right]\mathrm{d}l\end{aligned} \tag{21}$$

The variation of L_N is:

$$\begin{aligned}\delta L_N &= \int_0^L N\left(\xi'\delta\xi' + \eta'\delta\eta'\right)\mathrm{d}l + \int_0^L \frac{N}{A}I_P\vartheta'\delta\vartheta'\mathrm{d}l + \\ &\quad - \int_0^L \frac{N}{A}S_x\left(\delta\xi'\vartheta' + \xi'\delta\vartheta'\right)\mathrm{d}l + \int_0^L \frac{N}{A}S_y\left(\delta\eta'\vartheta' + \eta'\delta\vartheta'\right)\mathrm{d}l\end{aligned} \tag{22}$$

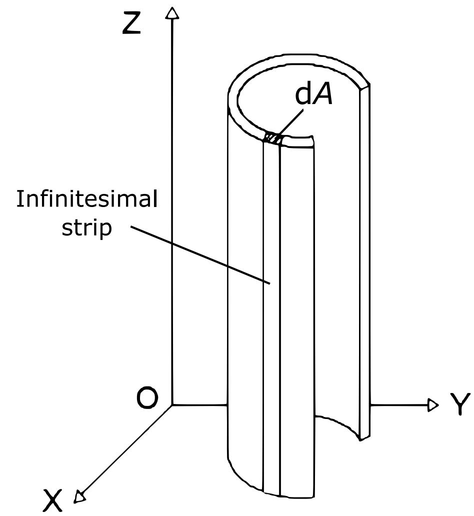

Fig. 3. Infinitesimal strip in a thin-walled open-section beam.

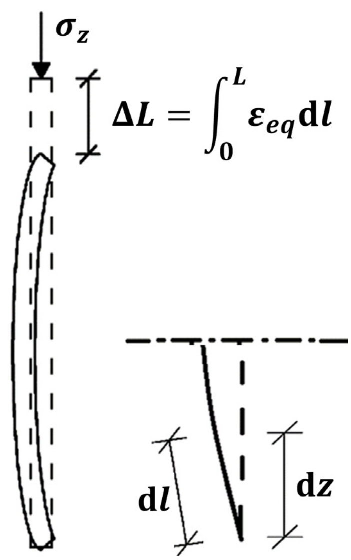

Fig. 4. Graphical representation of the term ΔL.

which integrated by parts provides:

$$\begin{aligned}\delta L_N = &\left[N\xi'\delta\xi + N\eta'\delta\eta + \frac{N}{A}\left(I_P\vartheta'\delta\vartheta - S_x\vartheta'\delta\xi - S_x\xi'\delta\vartheta + S_y\vartheta'\delta\eta + \right.\right. \\ &\left.\left. + S_y\eta'\delta\vartheta\right)\right]_0^L - \int_0^L \Bigg[N\xi''\delta\xi + N\eta''\delta\eta + \\ &+ \frac{N}{A}\left(I_P\vartheta''\delta\vartheta - S_x\vartheta''\delta\xi - S_x\xi''\delta\vartheta + S_y\vartheta''\delta\eta + S_y\eta''\delta\vartheta\right)\Bigg]\mathrm{d}l\end{aligned} \tag{23}$$

The work by the bending moment $\bar{M}_x$ is:

$$\begin{aligned}L_{\bar{M}_x} = \int_0^L \Bigg[&\int_A \frac{1}{2}\frac{\bar{M}_x}{I_{xx}}\left(y - y_G\right)\left(\xi'^2 + \eta'^2\right)\mathrm{d}A + \\ &+ \int_A \frac{1}{2}\frac{\bar{M}_x}{I_{xx}}\left(y - y_G\right)\left(y^2 + x^2\right)\vartheta'^2\mathrm{d}A + \\ &- \int_A \frac{\bar{M}_x}{I_{xx}}\left(y - y_G\right)\xi'\vartheta' y\mathrm{d}A + \int_A \frac{\bar{M}_x}{I_{xx}}\left(y - y_G\right)\eta'\vartheta' x\mathrm{d}A\Bigg]\mathrm{d}l\end{aligned} \tag{24}$$

If the following terms depending only on the geometry of the cross-section are defined:

$$\int_A \frac{y-y_G}{I_{xx}}\mathrm{d}A = a_x = 0 \tag{25a}$$

$$\int_A \frac{y-y_G}{I_{xx}}x\mathrm{d}A = b_x \tag{25b}$$

$$\int_A \frac{y-y_G}{I_{xx}}y\mathrm{d}A = c_x \tag{25c}$$

$$\int_A \frac{y-y_G}{I_{xx}}\left(y^2+x^2\right)\mathrm{d}A = d_x \tag{25d}$$

it is possible to write:

$$L_{\bar{M}_x} = \bar{M}_x \int_0^L \left[\frac{1}{2}d_x\vartheta'^2 - c_x\xi'\vartheta' + b_x\eta'\vartheta'\right]\mathrm{d}l \tag{26}$$

The variation of $L_{\bar{M}_x}$ is:

$$\delta L_{\bar{M}_x} = \bar{M}_x \int_0^L \left[d_x\vartheta'\delta\vartheta' - c_x\left(\xi'\delta\vartheta' + \vartheta'\delta\xi'\right) + b_x\left(\eta'\delta\vartheta' + \vartheta'\delta\eta'\right)\right]\mathrm{d}l \tag{27}$$

which integrated by parts provides:

$$\begin{aligned}\delta L_{\bar{M}_x} =& \bar{M}_x \left\{ \left[d_x\vartheta'\delta\vartheta - c_x\left(\vartheta'\delta\xi + \xi'\delta\vartheta\right) + b_x\left(\vartheta'\delta\eta + \eta'\delta\vartheta\right)\right]_0^L + \right.\\ &\left. - \int_0^L \left[d_x\vartheta''\delta\vartheta - c_x\left(\vartheta''\delta\xi + \xi''\delta\vartheta\right) + b_x\left(\vartheta''\delta\eta + \eta''\delta\vartheta\right)\right]\mathrm{d}l \right\}\end{aligned} \tag{28}$$

Eventually, the work by the bending moment $\bar{M}_y$ is:

$$\begin{aligned}L_{\bar{M}_y} =& \int_0^L \left[\int_A \frac{1}{2}\frac{\bar{M}_y}{I_{yy}}\left(x-x_G\right)\left(\xi'^2+\eta'^2\right)\mathrm{d}A + \right.\\ &+ \int_A \frac{1}{2}\frac{\bar{M}_y}{I_{yy}}\left(x-x_G\right)\left(y^2+x^2\right)\vartheta'^2\mathrm{d}A + \\ &\left. - \int_A \frac{\bar{M}_y}{I_{yy}}\left(x-x_G\right)\xi'\vartheta' y\mathrm{d}A + \int_A \frac{\bar{M}_y}{I_{yy}}\left(x-x_G\right)\eta'\vartheta' x\mathrm{d}A \right]\mathrm{d}l\end{aligned} \tag{29}$$

If the following terms depending only on the geometry of the cross-section are defined:

$$\int_A \frac{x-x_G}{I_{yy}}\mathrm{d}A = a_y = 0 \tag{30a}$$

$$\int_A \frac{x-x_G}{I_{yy}}x\mathrm{d}A = b_y \tag{30b}$$

$$\int_A \frac{x-x_G}{I_{yy}}y\mathrm{d}A = c_y \tag{30c}$$

$$\int_A \frac{x-x_G}{I_{yy}}\left(y^2+x^2\right)\mathrm{d}A = d_y \tag{30d}$$

it is possible to write:

$$L_{\bar{M}_y} = \bar{M}_y \int_0^L \left[\frac{1}{2}d_y\vartheta'^2 - c_y\xi'\vartheta' + b_y\eta'\vartheta'\right]\mathrm{d}l \tag{31}$$

The variation of $L_{\bar{M}_y}$ is:

$$\delta L_{\bar{M}_y} = \bar{M}_y \int_0^L \left[d_y\vartheta'\delta\vartheta' - c_y\left(\xi'\delta\vartheta' + \vartheta'\delta\xi'\right) + b_y\left(\eta'\delta\vartheta' + \vartheta'\delta\eta'\right)\right]\mathrm{d}l \tag{32}$$

which integrated by parts provides:

$$\begin{aligned}\delta L_{\bar{M}_y} =& \bar{M}_y \left\{ \left[d_y\vartheta'\delta\vartheta - c_y\left(\vartheta'\delta\xi + \xi'\delta\vartheta\right) - b_y\left(\vartheta'\delta\eta + \eta'\delta\vartheta\right)\right]_0^L + \right.\\ &\left. + \int_0^L \left[d_y\vartheta''\delta\vartheta + c_y\left(\vartheta''\delta\xi + \xi''\delta\vartheta\right) - b_y\left(\vartheta''\delta\eta + \eta''\delta\vartheta\right)\right]\mathrm{d}l \right\}\end{aligned} \tag{33}$$

Adding all the variations δL_N, $\delta L_{\bar{M}_x}$, and $\delta L_{\bar{M}_y}$:

$$\begin{aligned}\delta L_e =& \delta L_N + \delta L_{\bar{M}_x} + \delta L_{\bar{M}_y} = \\ =& \left[N\xi'\delta\xi + N\eta'\delta\eta + \right.\\ &\left. + \frac{N}{A}\left(I_P\vartheta'\delta\vartheta - S_x\vartheta'\delta\xi - S_x\xi'\delta\vartheta + S_y\vartheta'\delta\eta + S_y\eta'\delta\vartheta\right) \right]_0^L + \\ &+ \bar{M}_x\left\{\left[d_x\vartheta'\delta\vartheta - c_x\left(\vartheta'\delta\xi + \xi'\delta\vartheta\right) + b_x\left(\vartheta'\delta\eta + \eta'\delta\vartheta\right)\right]_0^L\right\} + \\ &+ \bar{M}_y\left\{\left[d_y\vartheta'\delta\vartheta + c_y\left(\vartheta'\delta\xi + \xi'\delta\vartheta\right) - b_y\left(\vartheta'\delta\eta + \eta'\delta\vartheta\right)\right]_0^L\right\} + \\ &- \int_0^L \left[N\xi''\delta\xi + N\eta''\delta\eta + \right.\\ &\left. + \frac{N}{A}\left(I_P\vartheta''\delta\vartheta - S_x\vartheta''\delta\xi - S_x\xi''\delta\vartheta + S_y\vartheta''\delta\eta + S_y\eta''\delta\vartheta\right) \right]\mathrm{d}l + \\ &- \bar{M}_x\left\{\int_0^L \left[d_x\vartheta''\delta\vartheta - c_x\left(\vartheta''\delta\xi + \xi''\delta\vartheta\right) + b_x\left(\vartheta''\delta\eta + \eta''\delta\vartheta\right)\right]\mathrm{d}l\right\} + \\ &+ \bar{M}_y\left\{\int_0^L \left[d_y\vartheta''\delta\vartheta + c_y\left(\vartheta''\delta\xi + \xi''\delta\vartheta\right) - b_y\left(\vartheta''\delta\eta + \eta''\delta\vartheta\right)\right]\mathrm{d}l\right\}\end{aligned} \tag{34}$$

The work by the distributed external loads p_x, p_y, and m_z, is:

$$L_{distr} = \int_0^L \left(p_x\xi + p_y\eta + m_z\vartheta\right)\mathrm{d}l \tag{35}$$

and the related variation is:

$$\delta L_{distr} = \int_0^L \left(p_x\delta\xi + p_y\delta\eta + m_z\delta\vartheta\right)\mathrm{d}l \tag{36}$$

The variation of the total potential energy of the beam can be determined and set equal to zero:

$$\delta W = \delta\Phi - \delta L_e - \delta L_{distr} = 0 \tag{37}$$

By inserting Eqs. (7), (34), and (36) into Eq. (37), two equations are obtained which must be satisfied. The first equation contains the finite terms and is verified at the end points of the beam (constrained points), that is for $z = 0, L$:

$$\begin{aligned}&\left[\frac{1}{2}N\left(y_G\delta\eta' + x_G\delta\xi'\right) - EI_{yy}\left(\xi''\delta\xi' - \xi'''\delta\xi\right) - EI_{xx}\left(\eta''\delta\eta' - \eta'''\delta\eta\right) + \right.\\ &\left. - EI_{\omega\omega}\left(\vartheta''\delta\vartheta' - \vartheta'''\delta\vartheta\right) + GI_t\vartheta'\delta\vartheta \right]_0^L + \\ &+ \left[N\xi'\delta\xi + N\eta'\delta\eta + \right.\\ &\left. + \frac{N}{A}\left(I_P\vartheta'\delta\vartheta - S_x\vartheta'\delta\xi - S_x\xi'\delta\vartheta + S_y\vartheta'\delta\eta + S_y\eta'\delta\vartheta\right) \right]_0^L + \\ &+ \bar{M}_x\left\{\left[d_x\vartheta'\delta\vartheta - c_x\left(\vartheta'\delta\xi + \xi'\delta\vartheta\right) + b_x\left(\vartheta'\delta\eta + \eta'\delta\vartheta\right)\right]_0^L\right\} + \\ &+ \bar{M}_y\left\{\left[d_y\vartheta'\delta\vartheta + c_y\left(\vartheta'\delta\xi + \xi'\delta\vartheta\right) - b_y\left(\vartheta'\delta\eta + \eta'\delta\vartheta\right)\right]_0^L\right\} = 0\end{aligned} \tag{38}$$

$$[N]=\begin{bmatrix} N & 0 & -N\frac{S_x}{A}-\bar{M}_x c_x-\bar{M}_y c_y \\ 0 & N & N\frac{S_y}{A}+\bar{M}_x b_x+\bar{M}_y b_y \\ -N\frac{S_x}{A}-\bar{M}_x c_x-\bar{M}_y c_y & N\frac{S_y}{A}+\bar{M}_x b_x+\bar{M}_y b_y & N\frac{I_P}{A}+\bar{M}_x d_x+\bar{M}_y d_y \end{bmatrix} \tag{42c}$$

Box I.

On the other hand, the second equation is integral:

$$\begin{aligned}&\int_0^L\left(EI_{yy}\xi^{IV}\delta\xi+EI_{xx}\eta^{IV}\delta\eta+EI_{\omega\omega}\vartheta^{IV}\delta\vartheta-GI_t\vartheta''\delta\vartheta\right)\mathrm{d}l+\\&+\int_0^L\Big[N\xi''\delta\xi+N\eta''\delta\eta+\\&+\frac{N}{A}\left(I_P\vartheta''\delta\vartheta-S_x\vartheta''\delta\xi-S_x\xi''\delta\vartheta+S_y\vartheta''\delta\eta+S_y\eta''\delta\vartheta\right)\Big]\,\mathrm{d}l+\\&-\bar{M}_x\left\{\int_0^L\left[d_x\vartheta''\delta\vartheta-c_x\left(\vartheta''\delta\xi+\xi''\delta\vartheta\right)+b_x\left(\vartheta''\delta\eta+\eta''\delta\vartheta\right)\right]\mathrm{d}l\right\}+\\&+\bar{M}_y\left\{\int_0^L\left[d_y\vartheta''\delta\vartheta+c_y\left(\vartheta''\delta\xi+\xi''\delta\vartheta\right)-b_y\left(\vartheta''\delta\eta+\eta''\delta\vartheta\right)\right]\mathrm{d}l\right\}+\\&-\int_0^L\left(p_x\delta\xi+p_y\delta\eta+m_z\delta\vartheta\right)\mathrm{d}l=0\end{aligned} \tag{39}$$

By separating the terms of Eq. (39) in relation to their variation $\delta\xi$, $\delta\eta$, or $\delta\vartheta$, it is possible to write a system of three equations in three unknowns:

$$EI_{yy}\xi^{IV}+N\xi''-\frac{N}{A}S_x\vartheta''-\bar{M}_x c_x\vartheta''-\bar{M}_y c_y\vartheta''=p_x \tag{40a}$$

$$EI_{xx}\eta^{IV}+N\eta''+\frac{N}{A}S_y\vartheta''+\bar{M}_x b_x\vartheta''+\bar{M}_y b_y\vartheta''=p_y \tag{40b}$$

$$\begin{aligned}&EI_{\omega\omega}\vartheta^{IV}-GI_t\vartheta''+\frac{N}{A}I_P\vartheta''-\frac{N}{A}S_x\xi''+\frac{N}{A}S_y\eta''+\bar{M}_x d_x\vartheta''+\\&-\bar{M}_x c_x\xi''+\bar{M}_x b_x\eta''+\bar{M}_y d_y\vartheta''-\bar{M}_y c_y\xi''+\bar{M}_y b_y\eta''=m_z\end{aligned} \tag{40c}$$

The previous system of three equations can be rewritten in a compact form:

$$E\,[I]\left\{\delta^{IV}\right\}+G\left[I_t\right]\left\{\delta''\right\}+[N]\left\{\delta''\right\}=\{F\} \tag{41}$$

where the matrices are defined as follows (see Box I for Eq. 42c):

$$[I]=\begin{bmatrix} I_{yy} & 0 & 0 \\ 0 & I_{xx} & 0 \\ 0 & 0 & I_{\omega\omega} \end{bmatrix} \tag{42a}$$

$$\left[I_t\right]=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_t \end{bmatrix} \tag{42b}$$

$$\{F\}=\begin{Bmatrix} p_x \\ p_y \\ m_z \end{Bmatrix} \tag{42d}$$

$$\{\delta\}=\begin{Bmatrix} \xi \\ \eta \\ \vartheta \end{Bmatrix} \tag{42e}$$

Matrix $[I]$ is the tensor of the moments of inertia, whereas matrix $[I_t]$ contains only the term related to the torsional stiffness factor and describes the primary torsion according to Saint Venant's Theory. Matrix $[N]$ contains the values of concentrated loads acting at the ends of the beam (these terms can be interpreted as the eigenvalues of the problem) multiplied by the geometric coefficients that describe the cross-section of the beam. The $\{F\}$ and $\{\delta\}$ vectors contain the values of the transverse distributed external loads and of the generalized displacements of the beam, respectively. In conclusion, by imposing the boundary conditions, it is possible to define the values of the critical loads that determine the instability of the beam. For each of the two ends of the beam, it is possible to identify six boundary conditions which may be static or kinematic. In particular, depending on the type of constraint, we can impose:

$$Clamped\ end \Longrightarrow \begin{matrix} \xi=0 \\ \eta=0 \\ \vartheta=0 \end{matrix} \quad ; \quad \begin{matrix} \xi'=0 \\ \eta'=0 \\ \vartheta'=0 \end{matrix} \tag{43a}$$

$$Hinged\ end \Longrightarrow \begin{matrix} \xi=0 \\ \eta=0 \\ \vartheta=0 \end{matrix} \quad ; \quad \begin{matrix} \xi''=0 \\ \eta''=0 \\ \vartheta''=0 \end{matrix} \tag{43b}$$

$$Free\ end \Longrightarrow \begin{matrix} \xi''=0 \\ \eta''=0 \\ \vartheta''=0 \end{matrix} \quad ; \quad \begin{matrix} \xi'''=0 \\ \eta'''=0 \\ EI_{\omega\omega}\vartheta'''-GI_t\vartheta'=0 \end{matrix} \tag{43c}$$

4. Equation of the non-uniform torsion with lateral–torsional buckling effects

The equations written so far are absolutely general. We intend now to emphasize two particular cases due to geometry and/or loading conditions. Consider a thin-walled open-section beam without the loads concentrated at its ends, that is $N=\bar{M}_x=\bar{M}_y=0$. Eq. (41) takes the following form:

$$\begin{bmatrix} EI_{yy} & 0 & 0 \\ 0 & EI_{xx} & 0 \\ 0 & 0 & EI_{\omega\omega} \end{bmatrix}\begin{Bmatrix} \xi^{IV} \\ \eta^{IV} \\ \vartheta^{IV} \end{Bmatrix}+\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -GI_t \end{bmatrix}\begin{Bmatrix} \xi'' \\ \eta'' \\ \vartheta'' \end{Bmatrix}=\begin{Bmatrix} p_x \\ p_y \\ m_z \end{Bmatrix} \tag{44}$$

The first two scalar equations are the usual equations of the elastic line, whereas the third,

$$EI_{\omega\omega}\vartheta^{IV}-GI_t\vartheta''=m_z \tag{45}$$

is the equation of the non-uniform torsion obtained by the variational energy method, and not by the static method proposed by Vlasov.

Consider now that the only load acting on the beam is the bending moment $\bar{M}_x$ ($N=\bar{M}_y=p_x=p_y=m_z=0$). Eq. (41) takes the following form:

$$\begin{aligned}&\begin{bmatrix} EI_{yy} & 0 & 0 \\ 0 & EI_{xx} & 0 \\ 0 & 0 & EI_{\omega\omega} \end{bmatrix}\begin{Bmatrix} \xi^{IV} \\ \eta^{IV} \\ \vartheta^{IV} \end{Bmatrix}+\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -GI_t \end{bmatrix}\begin{Bmatrix} \xi'' \\ \eta'' \\ \vartheta'' \end{Bmatrix}+\\&+\begin{bmatrix} 0 & 0 & -\bar{M}_x \\ 0 & 0 & 0 \\ -\bar{M}_x & 0 & 0 \end{bmatrix}\begin{Bmatrix} \xi'' \\ \eta'' \\ \vartheta'' \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}\end{aligned} \tag{46}$$

Considering that:

$$\xi''=-\frac{\bar{M}_x}{EI_{yy}}\vartheta \tag{47}$$

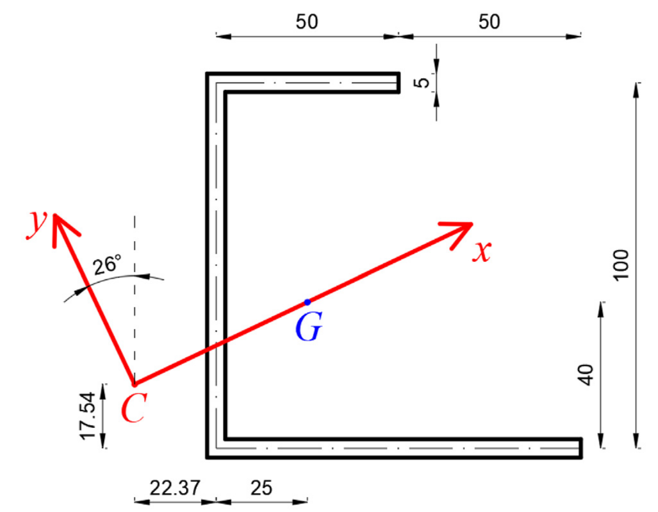

Fig. 5. Cross-section of the beam, C and G indicate the shear center and the gravity center, respectively (dimensions in mm).

the third equation of system (46) can be written as:

$$EI_{\omega\omega}\vartheta^{IV} - GI_t\vartheta'' - \frac{\bar{M}_x^2}{EI_{yy}}\vartheta = 0 \tag{48}$$

and represents the equation of the non-uniform torsion with lateral–torsional buckling effects. This equation can also be obtained conceptually, considering the overlapping of the effects of the non-uniform torsion Eq. (45) and of the equation of the lateral–torsional instability by Prandtl:

$$\vartheta^{IV} + \frac{\bar{M}_x^2}{EGI_{yy}I_t}\vartheta'' = 0 \tag{49}$$

In the present section, it has been shown how the remarkable cases present in the literature (Euler's problem, Prandtl's problem, Non-uniform Torsion theory) are all included within Eq. (41), which proves to be very general.

5. Numerical example

The results obtained investigating the stability of a simple cantilever beam are illustrated in the present section. Observe that the cross-section geometry does not present any symmetry, Fig. 5. Based on the geometrical and mechanical characteristics of the beam, which are shown in Table 1, the beam stability domain is obtained. Since it is a cantilever beam, the boundary conditions are:

$$z = 0 \left(Clamped\ end\right) \Longrightarrow \begin{array}{c} \xi = 0 \\ \eta = 0 \\ \vartheta = 0 \end{array} \quad ; \quad \begin{array}{c} \xi' = 0 \\ \eta' = 0 \\ \vartheta' = 0 \end{array} \tag{50a}$$

$$z = L \left(Free\ end\right) \Longrightarrow \begin{array}{c} \xi'' = 0 \\ \eta'' = 0 \\ \vartheta'' = 0 \end{array} \quad ; \quad \begin{array}{l} \xi''' = 0 \\ \eta''' = 0 \\ EI_{\omega\omega}\vartheta''' - GI_t\vartheta' = 0 \end{array} \tag{50b}$$

The stability domain of the beam is a function of the axial force N, and of the bending moments M_x and M_y. They represent the eigenvalues of the stability problem, as shown in Eq. (41). Since the computational solution of the system of Eqs. (41) is rather complex, only the intersections of the stability domain with the $N - M_x$ and $N - M_y$ planes are determined. This result is obtained by imposing in Eq. (41) $M_y = 0$ (to obtain the $N - M_x$ domain) or $M_x = 0$ (to obtain the $N - M_y$ domain). Then, using the Mathworks Matlab software, the

Table 1
Geometrical and mechanical characteristics of the beam.

Description	Symbol	Magnitude	Unit
Beam length	L	3,000	mm
Beam thickness	t	5	mm
Normal elastic modulus	E	2.1×10^5	N/mm^2
Shear elastic modulus	G	8×10^4	N/mm^2
Cross-section area	A	1,250	mm^2
Moment of inertia with respect to the x-axis	I_{xx}	2,454,873	mm^4
Moment of inertia with respect to the y-axis	I_{yy}	4,243,609	mm^4
Sectorial moment of inertia	$I_{\omega\omega}$	8.4×10^8	mm^6

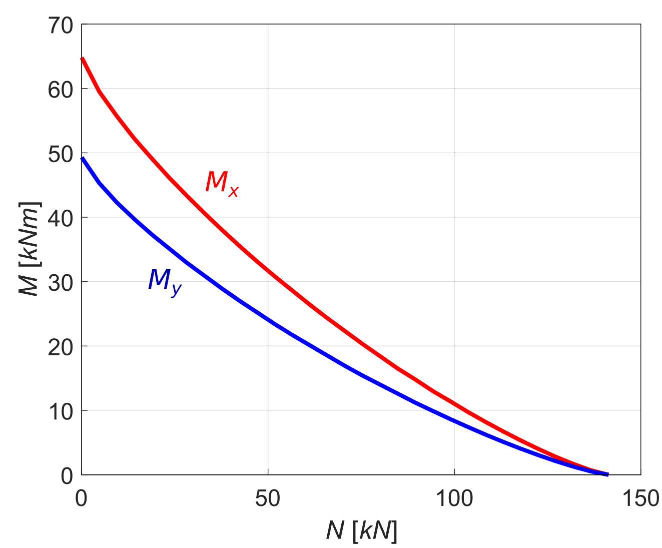

Fig. 6. Buckling domain of the beam.

numerical solution of Eq. (41) is obtained. This solution is achieved in discrete form by imposing the values of N, varying between 0 and the value of the Euler critical load, and computing the value of M_x (or M_y) that satisfies the system of equations. The graphs of the buckling domain in the N vs M plane are shown in Fig. 6 . Using these graphs, and known the value of the acting axial force, it is possible to determine the corresponding value of the critical buckling bending moment that leads to collapse due to instability of the beam. As can be seen, if $N = 0$, we obtain the value of Prandtl's critical moment. On the other hand, if $N = N_{cr}$, the value of M is equal to zero. Finally, it is noted that the buckling domains described in this example provide the critical loads that determine the coupled buckling collapse of the beam. These values could be compared to the limit values that determine the plastic deformation of the material and consequently the plastic collapse of the beam, but this is out of the objectives of this paper.

6. Conclusions

In this paper, a complete and original analytical formulation is developed for the stability of thin-walled open-section beams. Such a complex three-dimensional formulation, based on Vlasov's Theory, turns out to be absolutely general and, therefore, can be used to evaluate the stability of a beam with any constraint typology and loading conditions. The compactness of the final expressions, made even more evident by the matrix formulation, allows us for a convenient use in the case of automatic computations. The formulation presented in this paper is complete in that the effect of axial force, bending moments, and, especially, bimoment are considered in calculating the elastic strain energy. Differently from papers published in the past years by other Authors concerning similar formulations, the equation of non-uniform torsion with lateral–torsional buckling effects is obtained. In this fourth-order differential equation, containing only even derivatives, Vlasov's theory (thin-walled open-sections beams) and Prandtl's theory (flexural–torsional buckling) are combined into a single equation. In addition, in this paper the formulation is written by using a reference system with the axes X and Y oriented according to the principal directions. This choice makes the equations simpler than those proposed by other Authors [24,32,33], who have preferred using a generic coordinate system.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] L. Euler, Sur la force des colonnes (In French), Mem. Acad. R. Des Sci. Et Belle Lett. 13 (1759).
[2] L. Prandtl, Kipp-erscheinungen, ein fall von instabilem elastischem gleichgewicht (In German), in: Inaugural Dissertation D. Phil. Fakultät, Sekt. IId, Ludwigs-Maximilians-Universität München, 1899.
[3] A. Michell, Elastic stability of long beams under transverse forces, Phil. Mag. 48 (1899) 298–309, http://dx.doi.org/10.1080/14786449908621336.
[4] H. Wagner, W. Pretscher, Verdrehung und knickung von offenen profilen (In German), Luftfahrtforschung 11 (1934) 174–180.
[5] H. Wagner, W. Pretscher, Torsion and buckling of open sections, NACA Tech. Memo. (784) (1936).
[6] V. Vlasov, Raschet rebristykh svodov-obolochek i balok iz tonkostennykh profilei na sovmestnoe deistvie izgiba i krucheniya (In Russian), Proekt I Stand. (8,9,10) (1936).
[7] P. Znamenskii, Obshchaya ustoishivost' dlinnukh otkrytykh profilei pri prodol'nom szhatii (In Russian), Tekhnika Vozdushn. Flot. 12 (1934) 174–180.
[8] V. Vlasov, Tonkostennyye Uprugie Sterzhni (in Russian), Stroiizdat, 1940.
[9] N. Boloban, Issledovanie dyuralyuminievykh profilei (In Russian), 1936.
[10] F. Bleich, H. Bleich, Biegung, drellung und knickung von staben aus dunnen wanden (In German), in: Vorbericht Zweiter Kongress Internationale Vereinigung Fur Bruckenbau Und Hochbau, Berlin, 1936.
[11] F. Bleich, Buckling Strength of Metal Structures, McGraw-Hill, 1952.
[12] S. Timoshenko, Theory of Bending, Torsion and Buckling of Thin-Walled Members of Open Cross-Section. Collected Papers of Stephen P. Timoshenko, McGraw-Hill, 1953.
[13] S. Timoshenko, J. Gere, Theory of Elastic Stability, McGraw-Hill, 1961.
[14] V. Vlasov, Thin-walled elastic beams, in: Israeli program for scientific translation, US Science Foundation, 1961.
[15] J. Anderson, N. Trahair, Stability of mono-symmetric beams and cantilever, J. Struct. Div. 98 (1972) 269–286, http://dx.doi.org/10.1061/JSDEAG.0003114.
[16] M. Attard, M. Bradford, Bifurcation experiments on mono-symmetric cantilevers, in: 12th Australian Conference on Mechanics of Structures and Material, 1990, pp. 207–213.
[17] A. Ghobarah, W. Tso, A nonlinear thin-walled beam theory, Int. J. Mech. Sci. 13 (1971) 1025–1038, http://dx.doi.org/10.1016/0020-7403(71)90025-7.
[18] T. Roberts, Second order strains and instability of thin-walled bars of open cross-section, Int. J. Mech. Sci. 13 (1981) 297–306, http://dx.doi.org/10.1016/0020-7403(81)90033-3.
[19] Y. Yang, W. McGuire, Stiffness matrix for geometric nonlinear analysis, J. Struct. Eng. 112 (1986) 853–877.
[20] S. Kitipornchai, S. Chan, Nonlinear finite element analysis of angle and T beam-columns, J. Struct. Eng. 113 (1987) 721–739.
[21] A. Van der Heijden, W.T. Koiter'S Elastic Stability of Solids and Structures, Cambridge University Press, 2009.
[22] W. Koiter, Over de Stabiliteit Van Hetelastisch Evenwicht (in Dutch) (Ph.D. thesis), H. J. Paris, Amsterdam, English translation as NASA TT F- 10, 833 (1967) and AFFDL Report TR 70-25 (1970), 1945.
[23] W. Koiter, Theory of Elastic Stability, McGraw-Hill, 1979.
[24] M. Pignataro, A. Luongo, N. Rizzi, On the effect of the local overall interaction on the postbuckling of uniformly compressed channels, Thin-Walled Struct. 3 (1985) 293–321, http://dx.doi.org/10.1016/0263-8231(85)90009-6.
[25] M. Pignataro, A. Luongo, Asymmetric interactive buckling of thin-walled columns with initial imperfections, Thin-Walled Struct. 5 (1987) 365–382, http://dx.doi.org/10.1016/0263-8231(87)90027-9.
[26] Y. Pi, N. Trahair, S. Rajasekaran, Energy equation for beam lateral buckling, J. Struct. Eng. 118 (1992) 1462–1479, http://dx.doi.org/10.1061/(ASCE)0733-9445(1992)118:6(1462).
[27] N. Trahair, Flexural-Torsional Buckling of Structures, CRC Press, 1993.
[28] H. Ronagh, M. Bradford, M. Attard, Nonlinear analysis of thin-walled members of variable cross-section, part I: Theory, Comput. Struct. 77 (2000) 285–299, http://dx.doi.org/10.1016/S0045-7949(99)00223-0.
[29] H. Ronagh, M. Bradford, M. Attard, Nonlinear analysis of thin-walled members of variable cross-section, part II: Application, Comput. Struct. 77 (2000) 301–313, http://dx.doi.org/10.1016/S0045-7949(99)00224-2.
[30] F. Mohri, F. Brouki, J. Roth, Flexural-torsional post-buckling analysis of thin-walled elements with open sections, Thin-Walled Struct. 39 (2001) 907–938, http://dx.doi.org/10.1016/S0263-8231(01)00038-6.
[31] G. Taig, G. Ranzi, A. Luongo, GBT pre-buckling and buckling analyses of thin-walled members under axial and transverse loads, Contin. Mech. Thermodyn. 28 (2016) 41–66, http://dx.doi.org/10.1007/s00161-014-0399-9.
[32] G. Piccardo, A. Ferrarotti, A. Luongo, Nonlinear generalized beam theory for open thin-walled members, Math. Mech. Solids 22 (2017) 1907–1921, http://dx.doi.org/10.1177/1081286516649990.
[33] G. Ranzi, A. Luongo, A new approach for thin-walled member analysis in the framework of GBT, Thin-Walled Struct. 49 (2011) 1404–1414, http://dx.doi.org/10.1016/j.tws.2011.06.008.
[34] G. Tong, L. Zhang, A new derivation of the buckling theory of thin-walled beams, in: International Conference of Advances in Steel Structures, Hong Kong, China, 2002, pp. 129–137, http://dx.doi.org/10.1016/B978-008044017-0/50014-7.
[35] L. Zhang, G.S. Tong, Elastic flexural-torsional buckling of thin-walled cantiliver, Thin-Walled Struct. 46 (2008) 27–37, http://dx.doi.org/10.1016/j.tws.2007.08.011.
[36] A. Carpinteri, Buckling instability of Vlasov thin-walled open-section beams (lecture 12), Course of Static and Dynamic Instability of Structures, Politecnico di Torino, 2022.
[37] G. Nitti, Static, Dynamic, and Stability Analysis of High-Rise Buildings (Ph.D. thesis), Politecnico di Torino, 2020.
[38] A. Carpinteri, G. Lacidogna, G. Nitti, Open and closed shear-walls in high-rise structural systems: Static and dynamic analysis, Curved Layer. Struct. 3 (2016) 154–171, http://dx.doi.org/10.1515/cls-2016-0013.
[39] M. Pignataro, N. Rizzi, A. Luongo, Stability, Bifurcation and Postcritical Behaviour of Elastic Structures, Elsevier Science, 1991.
[40] V. Vlasov, Kruchenie i ustoichivost' tonkostennykh otkrytykh profilei (in Russian), in: Stroitel'Naya Promyshlennost', (6–7) 1938.
[41] V. Vlasov, Kruchenie, ustoichivost' i kolebaniya tonkostennykh sterzhnei (in Russian), in: Prikladnaya Matematika I Mekhanika, 3, 1939.
[42] S. Timoshenko, Theory of bending, torsion and buckling of thin-walled members of open cross section, J. Franklin Inst. B 239 (1945) 201–361.
[43] G. Nitti, G. Lacidogna, A. Carpinteri, Tall buildings subjected to horizontal loading: Analysis of two case studies by an in-house numerical code, in: Proceedings of the 23rd Conference of the Italian Association of Theoretical and Applied Mechanics (AIMETA), Salerno, Italy, 2017, pp. 292–300, September 4-7.
[44] A. Carpinteri, Advanced Structural Mechanics, CRC Press, 2017.
[45] A. Gjelsvik, Theory of Thin Walled Bars, John Wiley & Sons, 1981.
[46] G. Lacidogna, Tall buildings: Secondary effects on the structural behaviour, Struct. Build. 6 (2017) 391–405, http://dx.doi.org/10.1680/jstbu.16.00058.
[47] G. Lacidogna, G. Nitti, D. Scaramozzino, A. Carpinteri, Diagrid system coupled with shear walls: Analytical investigation on the dynamical response in tall buildings, in: Proceedings of the 8th International Conference on Computational Methods in Structural Dynamics and Earthquake Engineering (COMPDYN), Streamed from Athens, Greece, 2021, pp. 1793–1802, http://dx.doi.org/10.7712/120121.8599.19204.
[48] G. Nitti, G. Lacidogna, A. Carpinteri, An analytical formulation to evaluate natural frequencies and mode shapes of high-rise buildings, Curved Layer. Struct. 8 (2021) 307–318, http://dx.doi.org/10.1515/cls-2021-0025.
[49] G. Nitti, G. Lacidogna, A. Carpinteri, Structural analysis of high-rise buildings under horizontal loads: A study on the Piedmont Region Headquarters Tower in Turin, Open Constr. Build. Technol. J. 13 (2019) 81–96, http://dx.doi.org/10.2174/1874836801913010081.
[50] G. Lacidogna, G. Nitti, D. Scaramozzino, A. Carpinteri, Diagrid systems coupled with closed- and open-section shear walls: Optimization of geometrical characteristics in tall buildings, Procedia Manuf. 44 (2020) 402–409, http://dx.doi.org/10.1016/j.promfg.2020.02.277.
[51] A. Carpinteri, Structural Mechanics Fundamentals, CRC Press, 2014.

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Corrigendum to: "Buckling instability of Vlasov thin-walled open-section beams: The Euler–Prandtl coupled problem" [Int. J. Non-Linear Mech. 154 (2023) 104432]

Alberto Carpinteri [a,b], Giuseppe Nitti [a,*]

[a] Politecnico di Torino, Department of Structural, Geotechnical and Building Engineering, Corso Duca degli Abruzzi 24, 10129, Torino, Italy
[b] Shantou University, Department of Civil and Environmental Engineering, Zhujiang (Pearl River) Professor of Guangdong Province, Shantou, PR China

The authors regret that there was a mistake in the published paper:

ERRATA:

$$\Delta L_{yz} \cong \left(1 - 1 + \frac{\varphi_x^2}{2}\right) \mathrm{d}l = \frac{1}{2}\varphi_x^2 \mathrm{d}l = \frac{1}{2}\xi'^2 \mathrm{d}l \tag{15a}$$

$$\Delta L_{xz} \cong \left(1 - 1 + \frac{\varphi_y^2}{2}\right) \mathrm{d}l = \frac{1}{2}\varphi_y^2 \mathrm{d}l = \frac{1}{2}\eta'^2 \mathrm{d}l \tag{15b}$$

CORRIGE:

$$\Delta L_{yz} \cong \left(1 - 1 + \frac{\varphi_x^2}{2}\right) \mathrm{d}l = \frac{1}{2}\varphi_x^2 \mathrm{d}l = \frac{1}{2}\eta'^2 \mathrm{d}l \tag{15a}$$

$$\Delta L_{xz} \cong \left(1 - 1 + \frac{\varphi_y^2}{2}\right) \mathrm{d}l = \frac{1}{2}\varphi_y^2 \mathrm{d}l = \frac{1}{2}\xi'^2 \mathrm{d}l \tag{15b}$$

The authors would like to apologise for any inconvenience caused.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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* Corresponding author.
E-mail address: giuseppe.nitti@polito.it (G. Nitti).

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